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A

Matheſis Enucleata:

7507

OR, THE

ELEMENTS OF THE MATHEMATICKS.

To which is Annexed,
An Introduction to *Specious Analyſis*,
or, ALGEBRA.

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in the Univerſity of *Altorf*.

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The AUTHOR's PREFACE TO THE READER;

Containing a SYNOPSIS of his Method.

I.

THAT *the Reader may the better apprehend our Deſign and Aim, we have thought fit to premiſe ſome things concerning the Methods, both general and particular, we make uſe of in the following Treatiſe. For as heretofore a ſort of a blind Deference to, and ſuperſtitious Veneration of Antiquity, and eſpecially of* Ariſtotle, *has hinder'd the Growth and Progreſs of Natural Philoſophy, which of late has made ſuch conſiderable Advances, ſince it has ventur'd to ſtand upon its own Bottom; to make new Additions to former Inventions; to eſſay new and unknown Objects; to ſubſtitute Things inſtead of Names; Certainties inſtead of Doubts; and Experience in the room of dull Credu-*

lities; not derogating, in the mean while, from the deſerved Praiſes of the Ingenious among the Ancients: So, without doubt, Mathematicks alſo, unleſs our Predeceſſors had imagined that it had long ago been brought to its utmoſt Perfection by Euclid, Archimedes, Apollonius, *and other Ingenious Ancients, would have arrived long ſince to a higher Pitch, and by this time have ſurpaſs'd thoſe Limits which now we admire its arrival to.*

II.

It is confeſs'd by all, that no Human Knowledge whatſoever can lay a more juſt Claim to an unſhaken Evidence and Certainty, or boaſt an higher Neceſſity of its Demonſtrations, or a greater Multitude of undeniable Truths, than the Mathematicks; and that thoſe Propoſitions we have, found out by Archimedes, *demonſtrated by* Euclid, Apollonius, *and others, are at the ſame time unqueſtionable, and altogether wonderful. But we may with Truth affirm, that moſt of their Propoſitions may either be diſpoſed in a better Order, or propounded eaſier, or demonſtrated more evidently and directly, or taught after a more ſhort and compendious Way, now at leaſt after they are already found out, and with a great deal of Pains demonſtrated by their firſt Inventors: and of this Opinion are ſeveral of the beſt and moſt celebrated Mathematicians of the preſent Age.*

III.

It is certain, Euclid *has demonſtrated ſeveral Propoſitions, (as* Prop. 2, 3, 20, 30, lib. I. 2, 5, 6, 10, 15, 28, 29, lib.III, *&c.) whoſe Truth to any attentive Perſon appears from the very Terms, more clearly and certainly, than the Truth of* Axiom 13. lib. 1. *which his Interpreters dare not admit without a Demonſtration. And tho' thoſe ſuperfluous Demonſtrations derogate nothing from the Certitude of the Thing, yet by an unneceſſary Increaſe of the Number of Propoſitions, and (which frequently follows thence) an inverting the Order of Things, they breed Tediouſneſs and Confuſion.*

IV.

IV.

There are none, unleſs thoſe who are bigotted to Antiquity, but muſt own that the Elements of Euclid *are deſtitute of a juſt and orderly Diſpoſition of Things. For to omit, that in the firſt Book there are handled ſeveral ſorts of Subjects, and a great Variety of Properties demonſtrated of them promiſcuouſly, without any reſpect to Similitude or Conveniency; there is this never to be excuſed,* viz. *That, after he has in the firſt Book deduced and demonſtrated ſome particular Affections of Magnitude, he proceeds, in the ſecond, to handle what are univerſal and common to any Quantity; then in the third and fourth, he contemplates the Circle and the Properties of Figures inſcrib'd in it, or circumſcribed about it; in the fifth again, he treats of the univerſal Doctrine of Ratio's and Proportions; and yet not ſo univerſally, but in the ſeventh again he is obliged to demonſtrate the ſame of Numbers particularly, which might have been done for all Quantities whatſoever, by one general Demonſtration.*

V.

Then as to the Method of Demonſtration uſed by the Ancients; it is true that it nicely regarded the Certainty of its Concluſions, nor would it admit any thing into its Demonſtrations, which was not either a firſt Principle, and ſo ſelf-evident, (called by them an Axiom*) or might not be ſuppoſed, beyond all Contradiction, poſſible to be effected, (and on that account named a* Poſtulate*;) or, thirdly, an arbitrary Denomination of the Thing propoſed, which needed no Demonſtration, (and was called a Definition or Explication of a Term;) or, laſtly, which had not been demonſtrated evidently before: Yet, I believe, none will deny, but that this Method would have been more deſervedly eſteem'd, if with the certainty of its Concluſions there had been joined a greater Eaſineſs, Brevity, and Evidence, which is wanting in moſt of the Demonſtrations of the Ancients; who thought it enough, firmly and infallibly to eſtabliſh the Truth of their Theorems, and extort the Aſſent of their Readers, little regarding by what round-about Ways, by how many circumambulatory Propoſitions, and almoſt whole*

Volumes, it was done; that thereby they might be forced to acknowledge the thing to be ſo, when, at the ſame time, how it came to be ſo, or from what intrinſick Cauſe or Condition of the Subject requiring it, ſuch and ſuch an Attribute agreed to it, remain'd in the mean while obſcure, or altogether unknown.

VI.

Hence they made ſuch frequent uſe of Apagogical Demonſtrations, or Deductions ad abſurdum & impoſſibile, *which ought not to be done, but where no oſtenſive Demonſtration can be had, or for illuſtrating negative Propoſitions rather than demonſtrating them: for the Method of Deduction* ad impoſſibile, *does not ſo much demonſtrate the Truth itſelf directly, as the conſequent Abſurdity of the oppoſite Suppoſition: whence it follows very indirectly, (tho' moſt certainly) that the Propoſition is true; when, in the mean time, the original Reaſon of its Truth remains altogether hid, and in the dark.*

VII.

But that we may not ſeem unjuſtly to reject the particular Method of the Ancients, made uſe of by Euclid, *as in* lib. 12. Prop. 2, 10, &c. *and by others, but eſpecially by* Archimedes, *who peculiarly addicted himſelf to it, (whence it has been by ſome called the* Archimedian *Method, for example; and by* Renaldinus, *the Method* per Exploſum Exceſſum atque Defectum:) *Beſides, its Deduction doubly* ad abſurdum, *whereon it always relies, it infers the Equality of two Magnitudes* A *and* B *by a far-fetch'd round-about way; ſhewing, that if* B *be ſuppoſed greater or leſs than* A, *from either Poſition there would follow an Abſurdity; and thence, as it were, begging the Equality by a new Inference, which, tho' it may paſs free from Suſpicion, yet it neither ought, nor can be admitted, without this Limitation:* viz. *That in comparing thoſe things whoſe Natures are capable of Equality, if it can be demonſtrated, the one is neither greater nor leſs than the other, we may thence juſtly infer that they are equal.*

VIII.

The Learned are now generally of Opinion, that, beſides that Synthetick Method, whereby the Ancients either oſtenſively deduced their Problems and Theorems from evident and common Principles, or apagogically demonſtrated them by Deductions ad abſurdum, *they alſo made uſe of a certain ſort of Analyſis, whereby they found out thoſe Theorems and Problems; and which, to raiſe the greater Admiration in their Readers, they afterwards ſtudiouſly conceal'd and kept to themſelves: Which Method is undoubtedly preferable to the other, as not only demonſtrating the Certainty of the Propoſition ſo found, but, at the ſame time, ſhewing the Invention of them too: and this is that Method which* Vieta, Harriot, *and* Des Cartes, *and their Followers, have not only brought to light in this laſt Age, but to a great degree of Perfection too; and whereof* Renaldinus, *in that vaſt Work of his, intitled,* Ars Analytica Mathematum, *has given us a large Treatiſe.*

IX.

There has appear'd moreover of late another particular Method[*] *invented by* Bonaventura Cavallerius, *which is called,* The Method of Indiviſibles; *whereby the moſt difficult and abſtruſe Problems of Geometry are found out, and demonſtrated with an incredible Eaſe: (which is the above-mention'd* Renaldinus's *Opinion of it,* lib. 1. Reſol. & Comp. *p.* 239.) *For to demonſtrate the Equality or Proportions of Figures and Bodies that may be compared with one another by this Method, he works after a way which ſeems to be more natural than any other, by ſuppoſing plain Figures to conſiſt of innumerable Lines, and Solids of innumerable Planes (called their Indiviſible Parts or Elements, becauſe the Lines are conceived without Breadth, and the Planes without any Thickneſs) and relying on this ſelf-evident* Axiom, *That if all the Indiviſibles of one Magnitude collectively taken, be equal or proportional to all the correſpondent Indiviſibles of another, taken ſeparately, each equal or proportional to each,*

[*] Dr. *Wallis* is of Opinion this is nothing but an Improvement of the Ancients Method of Exhauſtions.

then alſo thoſe Magnitudes will be equal or proportional among themſelves. Which Inference can be guilty of no Fallacy, nor liable to any Error, as long as thoſe Elements are taken and conceived in that Senſe their Authors deſign them; which is ſufficiently demonſtrated by Renaldinus, lib. 1. Compoſ. & Reſol. p. 245, & 306. *and alſo by* Honoratus Faber, *in his Synopſis,* p. 24. *and Dr.* Barrow, *in* Lect. Geom. *p.* 24, &c. *againſt* Tacquet *and the other Adverſaries of this Method.*

X.

There is another Method a-kin to this, which may be properly named Generative, very much followed by Faber *in his Synopſis, and* Barrow *in his* Lect. Geom. *whoſe Author* Renaldinus *tells us was* Guldinus, lib. cit. p. 253. *ſhewing at large its Rules and Foundations in the following Pages,* viz. *the Riſe of Lines from the Motions of Points; of plane and curvilinear Surfaces, from the determinate Progreſs or Rotation of a given Line; and of Solids, by the various Motions of various Surfaces: the Productions whereof are ſo repreſented to the Imagination, that the intrinſick Nature of the Magnitudes thence ariſing may become known, and their Properties and Affections may, from their Natures thus known, be eaſily and briefly deduced.*

XI.

Near a-kin to this Method of Cavallerius *is that other of* Infinite Progreſſions, *wherein having found a certain Progreſſion of like Parts circumſcribed about, or inſcribed in any given Magnitude, which may be continued by Biſection* ad infinitum; *and then at length (by virtue of the Doctrine of Exhauſtions, founded on* Prop. 1. lib. 10. Eucl.) *will terminate in the Magnitude itſelf; I ſay, wherein the Sum of thoſe infinite Terms, collected by Rules on purpoſe for the Addition of thoſe Progreſſions, and conſequently the Quantity or Proportion of the propoſed Magnitude to any other given one, may be expreſſed or defined. But this Termination of Figures infinitely circumſcribed or inſcribed in a Circle, not pleaſing* Renaldinus (*altho' his Diſſenſion ſeems only to conſiſt in Words*)

Words) he exhibits another Method like it, (which he peculiarly calls his own) built on twelve fundamental Theorems, and illuſtrated by ſeveral Examples, lib. 1. de Reſol. & Comp. p. 277, & ſeq.

XII.

Of late alſo, the moſt Ingenious Sir Iſaac Newton, *to demonſtrate his* Philoſophiæ Naturalis Principia Mathematica, lib. 1. ſect. 1. *premiſes ſome Lemma's of his Method of* Rationes primæ & ultimæ, *or naſcent and evaneſcent Quantities, thereby to avoid the Tedioușneſs of deducing long and perplex'd apagogical Demonſtrations after the Manner of the Ancients. For finding that his Demonſtrations might be very much contracted by the Method of Indiviſibles; and knowing at the ſame time, that that Method was ſcrupled by ſome, and thought not very Geometrical, he rather choſe to found his Method on the Sums and Proportions of Quantities, which he calls Naſcent and Evaneſcent, which performs the ſame as the Method of Indiviſibles, and may be more ſafely uſed; which he inculcates in theſe very Words, and others ſuch like, in* Schol. *of* Lemma 11. *and alſo anſwers ſeveral Objections which might ſeem to make againſt it.*

XIII.

But it would be in vain for us to attempt, in this place, to explain all and each of thoſe various Methods at length; having only propoſed to ourſelves, to demonſtrate the chief and principal Theorems and Inventions in Mathematicks, and to uſe ſometimes one of them, and ſometimes another, (having firſt demonſtrated their Foundations) according as we ſhall judge this or that of them, fitteſt to demonſtrate the thing in hand, and ſo ſhew the Reaſons and Uſe of each of them in the Proceſs of this Diſcourſe. And altho' H. Faber, *in his* Synopſis, p. 8. *inſinuates, that Algebraick Terms ought not to be made uſe of in Geometrical Demonſtrations, becauſe that Algebraick Method ſeems to be too difficult for young Beginners; yet we are of the quite contrary Opinion, (nay, we can ſcarce doubt but that that Ingenious Man would alſo agree with us herein, if he* *he ſaw the way we make uſe of thoſe Foundations of Algebra, which is only of the moſt ſimple and general Principles of it) eſpecially in this caſe, where the ſaid Method is by little and little inſtill'd with the Demonſtrations themſelves; and the literal Computations taught from their firſt Principles, than which nothing is more eaſy: And this is that which we deſign to do, and ſo uſe the Learner by degrees to this ſort of Demonſtration, thereby to prepare him the better for the Analytick Geometry of the Moderns, which is the higheſt* Apex *of Mathematicks. But we had rather our Reader ſhould himſelf find, than we trouble ourſelves any further to tell him here, how compendiouſly we demonſtrate the Propoſitions of Geometry, by the help of theſe Analytick Notes, without the tediouſneſs of a long Chain of Conſequences, which would be otherwiſe unavoidable.*

XIV.

After this Way we deſign to go through the following Scheme. 1. *We ſhall deduce many Propoſitions of* Euclid, Archimedes, *and* Apollonius, *from our Definitions, and the Generations of Magnitudes therein propoſed; as Corollaries neceſſarily flowing from them, and confirm'd only by an immediate and ſimple Conſequence.* 2. *We ſhall demonſtrate their chief Theorems (for the ſake of which they were forced to demonſtrate ſeveral others before-hand, the knowledge whereof, for their own ſake, was not ſo neceſſary or valuable) without any long Series of foregoing Propoſitions, or foreign Principles, from a few direct and intrinſick Principles of their own. Whence,* 3. *it will follow, that after this Method we ſhall propoſe things, and treat of them in a more natural Order, and firſt of all deliver what are moſt univerſal and common to all Quantities, and then deſcend in a more ſpecial manner to what regards Magnitude; and diſtribute and diſpoſe all according to certain general diſtinct Claſſes of the Things to be treated of, and their Affections. Hence alſo,* 4. *We deduce from thoſe univerſal Theorems, by way of Corollary, the Precepts of vulgar Arithmetick, and ſpecious Computation, which afterwards we make uſe of in particular Demonſtrations, after a very ſhort and compendious Way; and,*

and, for this very reaſon, ſome learned Men of the preſent Age are of opinion, that the Ancients often fell into that tedious and intricate Prolixity in their Demonſtrations, becauſe they would not acknowledge the great Affinity there was between Arithmetick and Geometry, taking particular care not to introduce the Terms and Operations of Arithmetick into Geometry; tho' at the ſame time they never ſcrupled to transfer the Names of Plane, Square, Cube, &c. *to Numbers.* 5. *Laſtly, Having firſt demonſtrated the firſt and fundamental Theorems of Elementary Geometry, we may ſafely build on them the* Praxes *of all kinds of Mathematical Arts, that are moſt uſeful and requiſite to ſeveral Exigencies of human Life; as* (1.) Trigonometry *both Plain and Spherical, the Conſtruction and Uſe of the Tables of Sines and Tangents.* (2.) *The* Conſtruction *of* Logarithms, *and a compendious Application of them to* Trigonometry*: And* (3.) *the fundamental Precepts of Algebra, or the Analytick Art; by the help whereof the Learner may at length arrive to the higher and more recluſe Parts of Geometry, and become Maſter thereof: Not to mention ſeveral Geometrical and Arithmetical Problems, which we have all along derived from ſeveral of our Theorems, by way of Corollary; whoſe Number being very much augmented, it may be, ſome other time, may make an Appendix of this Work.*

XV.

And thus when we ſhall have demonſtrated not only the chief Theorems of the ancient Mathematicians, omitting the unneceſſary Crowd of thoſe that are only ſubſidiary, but alſo have demonſtratively deduced the fundamental Precepts of the moſt neceſſary and uſeful Arts that flow from them, and that are abſtracted from Matter, as of Arithmetick, Trigonometry, and Algebra; I hope none will doubt but that in this little Volume we have exhibited, as it were, the Nucleus *or Kernel of the pure and genuine Mathematicks, (for thoſe other Sciences and Arts, which go by the Name of mix'd Mathematicks, are moſt of them Parts of Natural Philoſophy, from the Application of Mathematicks to the Phænomena of Nature)*

ture) and ſo may juſtly bear the Name of Matheſis Enucleata.

XVI.

Nor are we ignorant, neither ſhall we conceal what ſeveral Learned Men have both propoſed and already done, for removing thoſe Difficulties and Blemiſhes of the ancient Mathematical Methods we have juſt now mentioned. The late Admoniſhments of the anonymous Author of L'Art de Penſer, *no leſs ingeniouſly than modeſtly delivered,* Part 4. Chap. 9, 10. *of the ſaid Treatiſe, are ſufficiently known; as alſo the laudable Endeavours of* A. Tacquet, *and* Honoratus Faber, *and ſeveral others above-mention'd, for contracting, new-ordering, and more eaſily and directly demonſtrating the chief Geometrical Inventions of the Ancients. There are moreover extant, of a certain anonymous Author,* Elementa Geometrica novo ordine ac methodo ferè demonſtrata, *printed at* London *about 26 Years ago. There are alſo* F. Ignatius Gaſton Pardies Elemens de la Geometrie, *&c. tranſlated into* Latin, *after the Third Edition, by the Famous* Schmidtius, *Profeſſor at* Geneva: *As alſo* F. Mich. Mourgues, *of the Society of* Jeſus, *has publiſh'd a Treatiſe, intitled,* Nouveaux Elemens de Geometrie, abreges par des Methodes particulieres en moins de Cinquante Propoſitions, *&c. There are alſo ſeveral other Eſſays of reducing the Mathematicks into a better Order and Method, the Titles whereof we have only as yet ſeen; and even while theſe Papers were in the Preſs, there happen'd into our Hands a Treatiſe of* F. Lamy'*s, intitled,* Les Elemens de Geometrie, ou de la Meſure des Corps, *&c. printed at* Paris 1685. *So that we may only ſeem to ſome to do what has been done already, in endeavouring to ſhew our Reader a new and ſhorter Way to the Mathematicks.*

XVII.

But as none can blame James le Maire, *becauſe, after the happy Diſcovery of the* Magellanick *Paſſage from the* Atlantick *into the* Pacifick *Sea, he would needs yet endeavour to find another ſhorter, which he accordingly did; nor thoſe who*

who now-a-days conſult about finding one from theſe Parts of the World, by the North *to the* Eaſt-Indies *: So alſo, in an Affair of ſuch moment, that one or a few are not ſufficient to bring it to Perfection, if any one comes after, not only invited, but alſo aſſiſted by the ingenious Endeavours of thoſe who have gone before him, ſhall undertake to add to their Inventions, to help on the Buſineſs by his Advices, and ſhew what things are capable of being farther poliſh'd, and the Method how to perform it, doubtleſs he ought not to be blamed, nor accuſed of Arrogance, unleſs, at the ſame time, he endeavours to depreciate the Eſſays of others, and cry up his own as the only valuable; which is very far from our Deſign, as the Work itſelf will abundantly ſhew. Moreover, as the Senſes of Men are differently affected by different Objects, and their Palates have different Reliſhes of the ſame thing, according as they are differently prepar'd : So the ſame Truth takes and inſinuates itſelf more eaſily with one propoſed and demonſtrated after this way, more with another after that way; and we are ſo much the more likely to ſuit the different Genii of different Perſons, by how many more and different Methods and Ways we ſhew them, leading to the ſame End, of which every one may take that which he likes beſt.*

XVIII.

Therefore we publiſh, by the Divine Aſſiſtance, theſe our Endeavours alſo, after ſo many other ingenious and elaborate ones in the ſame kind, not doubting the Approbation of ſome of our Readers. This at leaſt we can experimentally affirm, That not a few of thoſe to whom theſe our Thoughts were partly publickly read in Lectures, and partly privately taught, (for they were only deſign'd for Learners) were not a little taken with the conciſe Brevity and Facility of the Demonſtrations ; ſo that we may reaſonably hope to be acceptable to thoſe, to whom either Time or ſufficient Force of Genius is wanting, to run over the vaſt Volumes of the ancient Mathematicians, and comprehend their prolix Demonſtrations, and long Series's of far-fetch'd Conſequences; and as for thoſe who have both Leiſure and Genius to do ſo, this may ſerve for an Encouragement towards it, viz. *That after they have gone through the* *chief Truths and Propoſitions they contain, demonſtrated in a more eaſy and ſhorter Way, they may ſo much the more confidently adventure upon thoſe celebrated and ingenious Treatiſes, from the reading whereof they were before deterr'd, by the Length and almoſt inſuperable Difficulty of their tedious and perplex'd Demonſtrations.*

XIX.

Being now about to enter upon the Matter itſelf, we will only further hint theſe few Things. 1. *Since our whole Deſign is for the Advantage of young Students, (which ought to be a Profeſſor's chief Care and Study) we muſt not omit the Explication of the moſt ſimple Terms; eſpecially, ſince we deſign to deduce ſome Corollaries immediately from them, which heretofore have unneceſſarily increas'd the Number of Propoſitions and Demonſtrations.* 2. *To encumber our Work as little as we can with Words, we have made uſe, eſpecially in our Analytick Calculus, of ſome Symbols, as* = *for Equality, as alſo of* □ *and* ▭ *for Square and Rectangle, and of* $\sqrt{}$ *the common radical Sign for the ſquare Root, with the Line* $\sqrt{}$ *on the top for connecting of Quantities together, the Root whereof is jointly taken;* $\sqrt[3]{}$ *for the Cube Root,* $\sqrt{\sqrt{}}$ *for the Biquadratick Root.*

Note; *If four Quantities ſtand thus,* AB : CD :: EF : GH, *or* $a : b :: c : d$, *theſe Quantities are proportional; and* $\overline{AB}^2$ *is the ſame as* □ AB; *alſo* AB × CD *is the ſame as* ▭ AB *into* CD.

XX.

That the Reader may at one view ſee the Contents of the following Treatiſe, we have thought fit to preſent him here with it, by way of Synopſis. It is divided in two Books:

I.

The firſt whereof contains the chief and moſt ſelect Propoſitions of Euclid's *Elements, of* Archimedes's *Treatiſes of the*

the Sphere and Cylinder, as alſo of the Dimenſion of the Circle, &c. *Wherein what theſe Authors have demonſtrated by a long and tedious Series of Conſequences, and for the moſt part indirectly, we have here endeavour'd to demonſtrate directly, and ſo that the Demonſtration of each Propoſition depends either on no other, and ſo is evident by its own Light, or on a very few of the foregoing ones.*

II.

After the ſame way, in the ſecond Book, we treat of the Conick Sections, *and demonſtrate the chief Properties of the* Conoid, Spheroid, Cycloid, Conchoid, *and* Spiral Lines, *which are extant either in* Apollonius *or* Archimedes, *and others; and what they have demonſtrated by long and tedious Proceſſes, we have here exhibited in a ſhort and eaſy Compendium. And that,*

III.

In ſuch a Method as does not ſo much require intent and ſevere Thinking, as a bare and eaſy Inſpection, and Application of the Principles of Specious Algebra, and Method of Indiviſibles. Which yet,

IV.

We don't barely ſuppoſe, and remit our Readers to other Books to learn (which would be too troubleſome;) but in the Proceſs of the Work itſelf, they are gradually, and as occaſion preſents, derived from their original Fountain, and firſt Principles.

V.

By the ſame Way alſo the moſt uſeful and neceſſary Mathematical Praxes *are laid down, under the Names of* Corollaries *and* Scholia, *the Conſtruction of the Tables of Sines and Tangents taught, the Original and Uſe of the Logarithms demonſtrated, and the Precepts both of Plain and Spherical Trigonometry, deduced from their firſt Principles,* &c.

VI. *The*

VI.

The Praxis alſo of pure Arithmetick, and of common Decimal, and (which is ſeldom uſed) of Tetractical; as alſo the Doctrine of Surds, are derived from their firſt Original. Whereunto,

VII.

As a Complement of the whole Work, we have added an Introduction to the Specious Analyſis, or new Geometry of the Moderns, particularly according to the Method of Des Cartes, *but much facilitated by later Inventions, and comprizing the Precepts of the Art in ſix or ſeven Pages, but illuſtrated with above forty Examples in the different Degrees of Equations.*

What the Reader's Opinion will be of theſe our Endeavours, deſign'd only for the Uſe of young Students, Time muſt teach us. The Author himſelf at leaſt, amongſt his other Performances, allows theſe the firſt place.

ERRATA.

Pag. 43. l. 7. r. $\frac{1}{2}ab$. P. 62. l. 1. r. $5\surd 2$. Ibid. l. 2. r. $3\surd 2$. P. 84. l. 27. r. $\frac{1}{2}a$. P. 101. l. 7. r. *thoſe*. P. 122. l. 13. dele *one*. P. 129. l. ult. r. *and as* DB *to* DE, *ſo* DE *to* EB. P. 224. l. 33. r. $AF = \frac{1}{2}a +$

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Matheſis Enucleata:
OR, THE
Elements of the Mathematicks.

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BOOK I.

Explaining the firſt Principles of the Mathematicks; among which are (in the firſt place) Definitions, *and ſome* Conſectaries *that flow from them.*

CHAP. I.

Containing the Definitions or Explications of the Terms which relate to the Object of Mathematicks.

DEFINITION I.

MATHEMATICKS is the Science or Knowledge of Quantity, and of Beings, as far as they are ſubject to it, or meaſurable, and may juſtly claim the Name of *Univerſal*, while it is employ'd in demonſtrating thoſe Properties which are common to all or moſt *Quantities*. But when it deſcends to the different Species of Quantity, and is buſied in contemplating the Affections belonging particularly to this or that Quantity, it is diſtinguiſhed by various Names, and diſtributed into various Parts, according to the various Diverſities of the Objects.

DEFINITION II.

QUantity may be defined, in general, to be whatever is capable of any ſort of Eſtimation or Menſuration, as immediately the Habitudes and Qualities of Things; for example, the Multitude of Stars in the Heaven, or of Soldiers in an Army, the Length of a Rope or Way, the Weight of a Stone, the Swiftneſs or Slowneſs of Motion, the Price of Commodities, &c. but mediately, the very Things themſelves wherein thoſe eſtimable Qualities are inherent. Whence, with the ingenious *Weigelius*, we may not incongruouſly reduce them all to theſe four Kinds or Genders, *viz.* 1. To *Natural Quantities*, or ſuch as Nature has furniſhed us with; as Matter with its Extenſion and Parts; the Powers and Forces of Natural Bodies; as Gravity, Motion, Place, Light, Opacity, Perſpicuity, Heat, Cold, &c. 2. To *Moral Quantities*, depending, for the moſt part, on the Manners of Men, and arbitrarious Determinations of the Will; as for example, the Values and Price of Things, the Dignity and Power of Perſons, the Good or Evil of Actions, Merits and Demerits, Rewards and Puniſhments, &c. 3. To *Notional Quantities*, ariſing from the Notions and Operations of the Underſtanding; as *e. g.* the Amplitude or Narrowneſs of our Conceptions, Univerſality, or Particularity, &c. in Logick; the Length or Brevity of Syllables, Accent, Tone, &c. in Grammar. And, laſtly, To *Tranſcendent Quantities*, ſuch as are obvious in Moral, Notional, and Natural Beings; as Duration, *i. e.* the Continuation of the Exiſtence of any Being; which in Phyſicks eſpecially is named *Time*, and may be conceived as a Line, &c. To theſe you may moreover add *Unity*, *Multitude* or *Number*, *Neceſſity* and *Contingency*.

DEFINITION III.

NUmber (whereon we ſhall make ſome ſpecial Remarks) if it be taken concretely, is nothing but an Aggregate or Multitude of any ſort of Beings; taken abſtractedly it is, as *Euclid* calls it, μονάδων ποσότης· a *Multitude*, or *Quotity of Unities*. On the one hand, Number, *i. e. many*, are oppoſed to one, and in that ſenſe Unity is not a Number; On the

other

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other hand, Unity may be eſteemed a Number, ſince it is ſome Quotity (if I may be allowed the Expreſſion) as well as two or three, *&c.* But as we denote or ſignify particular Things when we ſpeak of them univerſally, by the Letters of the Alphabet, A, B, C, (*a*, *b*, *c*,) *&c.* as univerſal Signs or Symbols of them; ſo for diſtinctly and compendiouſly expreſſing the innumerable Variety of Numbers, Men have found out various Notes, the moſt natural whereof are Points diſpoſed in particular extended Orders, as . . . to denote
. . .
Three . . . to denote Nine, *&c.* But that way which is
. . .
moſt commodious for Practice, is by the common Notation, or Cyphers, 1, 2, 3, 4, 5, 6, 7, 8, 9. the Invention whereof, as we have it by vulgar Tradition, is owing to the *Arabians*. By a very few of theſe we expreſs any Number, tho' never ſo great, by a wonderful, tho' now-a-days familiar Artifice; the firſt Inventor of them having eſtabliſhed this as an arbitrary Law, that the firſt of them ſhall ſignify *Unity*, or *one*; the ſecond *two*, &c. as often as they ſtand alone; but placed in a Row with others, or on the left hand of one or more 0's or Cyphers (which of themſelves ſtand for nothing, but fill up empty Places) if in the ſecond, before a Cypher, they denote Tens; if in the third, Hundreds; in the fourth, Thouſands; in the fifth, Tens of Thouſands; in the ſixth, ſo many Hundreds of Thouſands; in the ſeventh, ſo many Thouſands of Thouſands, or Millions; in the eighth, Tens of Millions, *&c.* and ſo onwards, increaſing always in a decuple Proportion, by Tens, Hundreds, Thouſands, *&c.*

COROLLARY I.

HENCE you have a way of exprſſing or writing any Sum by theſe Notes, or Figures, which you may have expreſſed in Words; as if we were to expreſs in Notes the Year of our Lord One Thouſand Six Hundred Ninety and Nine, it is manifeſt, that according to the Method above deſcribed, by placing nine on the right hand in the firſt place, and nine again in the ſecond towards the left, ſix in the third, and one or Unity in the fourth, the Buſineſs will be done. Thus it will be eaſy to any one, with a little Attention to expreſs any Number whatſoever by theſe Notes; as ſuppoſe Eleven Thouſand Eleven Hundred and Eleven, which is propoſed by *Swinter in Delic. Phyſico-Math. Part.* 1. *Prob.* 75.

COROLLARY II.

HENCE you have alſo the Foundation and Reaſon of the Rule of *Numeration* in Arithmetick, or expreſſing any Numbers or Sum in Words, which you ſee written in Figures; which for greater eaſe may be done thus, *viz.* beginning from the firſt Figure towards the right hand, over every fourth Figure note a Point (including always that which was laſt pointed) and at every ſecond Punctation or Point, draw ſhort Strokes thus ′, one over the ſecond, two over the fourth, *&c.* the firſt denoting Millions, the ſecond Millions of Millions, or Bimillions, the third Trimillions, *&c.* and the intercepted Points the Thouſands in their kinds, *&c.*

SCHOLIUM.

HERE I cannot omit, on this Occaſion, what the forementioned *Weigelius* has hinted about another way of *Numeration*, and which Dr. *Wallis* mentions, *Oper. Mathem. Part* 1. *p.* 25, and 66. ſhewing there a way (and illuſtrating it by Examples) of *Numeration*, and of expreſſing the Figures; which proceeds thus. Whereas now-a-days, in numbring, we aſcend from Unity or 1 to ten; (the Reaſon whereof, after which *Ariſtotle* makes a prolix Inquiry, *Probl.* 3. *Sect.* 15. was taken without doubt from our Fingers, being ten in Number) if from Unity we proceed only to four, (which *Ariſtotle*, in the ſame Place, tells us ſome of the *Thracians* uſed to do of old) and thence returning back again to Unity, we ſhould proceed again after the ſame way; we might after that way obtain a vaſtly more ſimple and eaſy Arithmetick than we have now-a-days. Which even hence we may conclude, becauſe for Multiplication and Diviſion there would need no other Table (or *Pythagorick Abacus*) than this eaſy and ſhort one:

1 . 1 . 1 once one is one;
2 . 2 . 10 *i. e.* twice two are four.
2 . 3 . 12 twice three are four and two.
3 . 3 . 21 thrice three are twice four and one.

And altho' it is pity that we can't hope now-a-days to ſubſtitute this vaſtly eaſier way of Computation, inſtead of the other now in uſe, becauſe the other is univerſally received, and moſt ſorts of Meaſures and other Quantities are fitted and accommodated to the decuple Proportion; yet it ought not to be altogether neglected in Mathematicks, which might receive very great Advantage hereby, eſpecially in Trigonometry,

metry, if the ingenious Invention of the Logarithms had not already ſupplied its Uſe therein. The whole Foundation of this *Quaternary Arithmetick* is placed only in theſe three Notes, or Figures, *viz.* 1,2,3; ſo that any one of them alone, or in the firſt place, ſhould denote Units, in the ſecond place Tetrads, or ſo many Fours (or Quaternions) in the third place ſo many Sixteens, in the fourth place ſo many times four Sixteens, or 64's, &c. always proceeding in a Quadruple Proportion. For which way of Numeration, there might be found out Terms as commodious as thoſe we now uſe, and which are thereby grown familiar to us; as one, ten, twenty, a hundred, a thouſand, &c. which will be evident by what follows:

One, *Unum*, ...	1 . .	One.
Ten, *Decem* . . .	,10 . .	*Quatuor*, *Tetras*, *a Quaternion* (*or* Four.
Twenty, *Viginti* . .	20 . .	*a Biquaternion.*
Thirty, *Triginti* . .	,30 . .	*a Triquaternion.*
Hundred, *Centum* .	100 . .	*a Tetraquaternion.*
Thouſand, *Mille*	1000 . .	*a Quartan.*
Ten Thouſand,	10000 &c.	*a Tetraquartan*, &c.

DEFINITION IV.

A *Magnitude* is whatever is conceived to be extended or continuous, or has Parts one without another, and contained within ſome common Bound or Bounds, wherein that is called a *Point* which is conceived (as indiviſible, or) to have no Parts, and ſo no Magnitude, but is notwithſtanding the Beginning or firſt Principle of all Magnitude.

DEFINITION V.

If we conceive a Point A, (*Fig.* 1.) to be moved towards B; by this Motion it will leave a Trace, or deſcribe the Magnitude AB of one only Dimenſion, that is, Length without Breadth, or which at leaſt we are to conceive ſo, and is called a *Line*: If that Line A,B be conceived again ſo to be moved as that its extreme Points A, B, ſhall deſcribe other Lines BC and AD, it will deſcribe by that Motion the Magnitude ABCD, or (to denote it more compendiouſly by the Diagonal Letters) AC or BD, having both Length and Breadth, but without any Depth or Thickneſs, or at leaſt ſo to be conceived, and this is called a *Superficies*, or *Surface*. Laſtly, if this Surface AC be conceived ſo to move, *e. g.* upwards or downwards, that its oppoſite Points A and C again deſcribe other Lines AF and CH, and conſequently

fig. 1. p. 10.

each of its Lines other Surfaces, &c. by this Motion there will be formed a Magnitude of three Dimenſions, which we call a *Solid* or *Body*, which we will alſo denote by the two diametrically oppoſite Letters AH and DG. But as this Motion of a Point, Line, or Surface may be various, ſo there will be produced by them various ſorts of Lines, Surfaces, and Solids: But theſe Productions ſtop here, and proceed no further; for the Motion of a Body can only produce another Body greater than the firſt, but no more new Dimenſions.

CONSECTARIES.

I. POINTS therefore being moved thro' equal Intervals in the ſame, or a like Way or Trace (*e. g.* in a ſtreight or the ſhorteſt Trace) deſcribe equal Lines: and

II. The ſame or equal Lines moved thro' the ſame right-lined or curvilinear Paths, deſcribe equal Surfaces; and

III. Equal Surfaces moved according to the ſame Methods and Conditions deſcribe equal Solids; which, if rightly underſtood, are the firſt, certain, and infallible Foundations of the Method of Indiviſibles. But here you muſt take care to diſtinguiſh between the way which the Line itſelf deſcribes, and that which its Ends or extreme Points deſcribe: For al- f. 24, p. 18. tho' *e.g.* the Point *a* (*Fig.* 24.) moves along in a more oblique way than A, and ſo deſcribes a longer Line *a c*; yet the Line *a b* deſcribes, by a parallel Motion, an equal Space with the Line AB, (*viz.*) the ſame which the whole Line A*b*, whereof they are Parts, would deſcribe. See *Faber's Synopſis, p. m.* 13.

DEFINITION VI.

BUT that we may a little further proſecute this Geneſis of Magnitudes (as very much conducing to underſtand f. 2. p. 10 their Nature and Properties) if the Point A (*Fig.* 2.) moves to B the ſhorteſt way, it deſcribes the right Line AB; but if in any (one) other, it will deſcribe the Curve or compounded Line ACB: From whence, with *F. Morgues*, we may infer theſe

CONSECTARIES.

(a) Eucl. Ax. 14.

I. THAT two Right Lines (*a*) beginning from the ſame Point A, and ending in the ſame Point B, will neceſſarily coincide, nor can they comprehend or incloſe Space; for if they did, one muſt deviate, and ſo would ceaſe to be a Right Line.

II. In a Space comprehended by three Right Lines AB, BC, CA, (*a*) any two taken together muſt needs be longer than any one alone. Moreover, we may add this beforehand, *viz*, that,

(*a*) Eucl. l. 1. Prop. 20.

III. In a Circle a Right Line drawn from A to B (*Fig.* 3.) will fall within the Circle, becauſe the Curve Line ADB deſcribed, as we ſhall hereafter ſhew, being longer than a Right Line, muſt neceſſarily fall beyond it, or on the Outſide of it. And laſtly, that

p. 10

IV. A Tangent, or Line (*b*), which does not cut or enter into the Circle, touches it only in one Point.

(*b*) Eucl. Prop. 2. lib. 3.

Moreover, if a Right Line AB (*Fig.* 4.) moves on another Right Line BC, being always in the ſame Poſition to itſelf, it will generate a *Plane Surface*; to which a Right Line being any way applied, will touch it with all its Point, as *Faber* rightly deſcribes it: If a Right Line be moved on a Curve, or a Curve on a Right Line, &c. they will generate a *Curve Surface*, called *Gibbous*, or *Convex* without, and *Concave* within.

p. 10

DEFINITION VII.

IF a Right Line be fixed at one of its Ends A, and the other End be moved round (*Fig.* 5.) it will deſcribe in this Motion a circular Plane, or a Circle; and by the Motion of its End or extreme Point B, the *Periphery* or *Circumference* of that Circle BEF. The fixed Point A is called the Center of that Circle, the Lines AB, AC, &c. its *Radii* or *Semi-Diameters*, all of which are equal one to another. Any Right Line BC, drawn thro' the Center from one Part of the Circumference to another, is called the *Diameter*, and divides the Circle into two Semicircles BEC and BFC. The Circumference of a Circle, whether great or ſmall, is divided into 360 equal Parts, called Degrees, and each Degree into 60 Minutes, &c. From this Generation of the Circle preſuppoſed, there evidently follow theſe

p. 10.

CONSECTARIES.

I. THAT two Circles, which cut one another, (*a*) cannot have the ſame common Center; for if they had, the *Radii* ED and EA drawn from the common Center E (*Fig.* 6.) would be equal to the common *Radius* EB, that is, the Part equal to the Whole.

(*a*) Eucl. 5. l. 3.

p. 10

II. Nor can two Circles, touching (*b*) one another inwardly, have one and the ſame Center, for the ſame Reaſon.

p. 10. (*b*) Eucl. 6. l. 3. (*c*) Eucl. 8. l. 3.

III. Of Lines falling from any given Point without the (*Fig.* 7.) Circle, and (*c*) paſſing thro' the Periphery to the oppoſite concave Part of it, that which paſſes thro' C the Center is the longeſt, *viz.* AB; and of the others that which is neareſt to AB is longer than that which is more remote: But, on the contrary, of thoſe which fall on the Convex Periphery, that which tends towards the Center, as A*b*, is the leaſt, and the reſt gradually greater, and there can be but two, as AE and AF, or A*e* and A*f*, equal: All which will appear very evident by drawing other Circles from the Center A thro B, D, E, and *b*, *d*, *e*. Or thus: Having drawn two other Circles, with the Radii AB and A*b*, if we conceive the Radii A*b* and C*b* to move towards the right Hand, their Ends will always recede further from one another; the ſame is alſo evident of the Radii AB and CB, moved alſo to the Right together.

p. 10. (*a*) Prop. 15. lib. 3.

IV. Moreover (*Fig.* 8.) of all Lines drawn within a Circle (*a*) the Diameter is the greateſt, and the reſt gradually leſs, by how much the more remote they are from the Center, &c. Which will be very evident to any one who contemplates a Circle inſcribed in a Square, as alſo the Geneſis of the Curvity itſelf; as alſo many other ways which I ſhall now omit: or to mention one more thus, Becauſe the two Radii CA and CB being moved, in order to meet together, neceſſarily approach nearer to one another in their extreme Points.

DEFINITION VIII.

p. 10.

THE Aperture or opening of two Lines (*Fig.* 9.) AB, AD, &c. that are both fixed at one end at A, and the other Ends opened or removed farther and farther from one another, is called an *Angle*, and uſually denoted by three Letters, D, A, B, (whereof that which denotes the Angular Point always ſtands in the middle) and is meaſured by the Arch of a Circle BD, or a certain Number of Degrees which it intercepts. The greateſt Aperture of all, *viz.* BAC, is when the two Legs of the Angle AB and AC make one Right Line, and is meaſured by a Semicircle, or 180 Degrees. The mean or middle Aperture BAE or CAE, when one Leg EA is erected on the other AB or AC at Right Angles, ſo that it inclines neither one way nor the other, (thence called a *Perpendicular*) is named a *Right Angle*, whoſe Meaſure is conſequently a Quadrant (or quarter Part) of a Circle or 90 Deg. Wherefore a Semicircle is the Meaſure of two Right Angles; an Aperture or Angle BAD leſs than a Right Angle (and ſo meaſured by leſs than 90 Degrees) is called an *Acute Angle*; and that which is greater than a Right Angle, as DAC (and ſo conſiſting of more than 90 Degrees) is called an *Obtuſe Angle*. Whence we may now draw theſe

COROLLARIES.

I. TWO or more contiguous Angles (*a*) conſtituted on the ſame Right Line BC, and at the ſame Point A (as DAB, and DAC, or DAB, DAE, and EAC) make two Right Angles, as filling the Semicircle; and conſequently,

(*a*) ***Eucl.* 13. *lib.* 1. *with the Coroll.***

II. All the Angles that can be conſtituted about the Point A (as filling the whole Circle) are equal to four Right ones: and contrariwiſe (*b*) if two Right Lines AB and AD meet on the ſame Point A of another Right Line AC, and make the contiguous Angles equal to two right ones, that is, if they fill a Semicircle, BC will neceſſariy be the Diameter of a Circle, and conſequently a Right Line.

(*b*) ***Euclid.* *Prop.* 14.**

III. If one of the Contiguous Angles BAE be a Right one, the other CAE will be ſo alſo.

IV. If two Right Lines AB, DE (*Fig.* 12.) cut one another in C, the four Angles they make will be equal to four Right ones. p. 10.

V. And as it is evident at firſt ſight (*Fig.* 10.) that any Circle having one half (or Semicircle) folded on the other, at the Diameter ECD, the two Semicircles EHD, and EID, muſt needs agree, or every where coincide one with the other; ſo if the Angle ACD be ſuppoſed equal to the Angle BCD; that is, the Arch AD to the Arch BD, having one Leg CK or CL common; the others AC and BC being ſuppoſed before equal: then, p. 10.

1. The Baſes BL and AL, KB and KA, will (*c*) be alſo equal, for theſe will coincide too, and therefore the Angles alſo.

(*c*) ***l.* 1. *prop.* 4. 8.**

2. The Line AB being biſected in K, the two Angles (*d*) at K, will alſo coincide and be equal, and conſequently Right Angles. And contrariwiſe,

(*d*) ***lib.* 3. 3.**

3. The

(e) *lib.* I. 5. 3. The Angles at the Baſe of the equal (*e*) Legs CAB, CBA, and alſo thoſe below the Baſe, the Legs being produced to F and G, are equal.

4. Conſequently the Spaces ACL and BCL, ACK and BCK, are equal to one another.

5. The Contiguous Angles AED and BED ſtanding on equal Arches, AD and BD are equal; and contrariwiſe, as alſo thoſe that are not contiguous, if their Vertexes are equidiſtant from E, *&c.*

6. It is hence alſo manifeſt, that a Perpendicular erected on the Middle of any Line AB, inſcribed in a Circle, paſſes through its Center, by what we have juſt now ſaid. And if you likewiſe erect Perpendiculars on the Middle of any two Lines, *ab* and *bm* (*Fig.* 11.) connecting any two Arches, or any three Points, *a*, *b*, *m*, that are not placed all in the ſame Right Line, thoſe two Perpendiculars *ke*, *no*, will determine (by their Interſection) the Center of a Circle that ſhall paſs through theſe three Points.

DEFINITION IX.

IF one Right Line DE cuts or paſſes thro' another AB, (*Fig.* 12.) the oppoſite Angles at the Interſection ACD and ECB are called *Vertical*; as alſo the other two ACE and DCB: Whence follow theſe

COROLLARIES.

(a) *Eucl. lib.* 1. 15. I. THat the Vertical Angles are always (*a*) equal; for both ACD and ECB with the third ACE, which is common to both, fill or are equal to a Semicircle; as likewiſe both ACE and DCB with the third ECB, which is common.

II. Contrariwiſe; if at (*b*) the Point C of the Right Line DE, the two oppoſite Lines AC and CB make (b) *The ſame Prop. Schol.* 1. the Vertical Angles X and Z equal, then will AC and CB make one Right Line; for, ſince X and O make a Semicircle, and Z and X are equal, by Hypoth. O and Z will alſo make or fill a Semicircle, whoſe Diameter will be ACB.

(c) *Schol.* 2. III. By the ſame Argument it will appear, that of four Lines (*c*) proceeding from the ſame Point, ſo as to make the oppoſite Vertical Angles equal, the two oppoſite ones AC and CB, as alſo DC and CE, will make each but one Right Line; for ſince all the four Angles together make a whole Circle, or four Right Angles, and the Sum of

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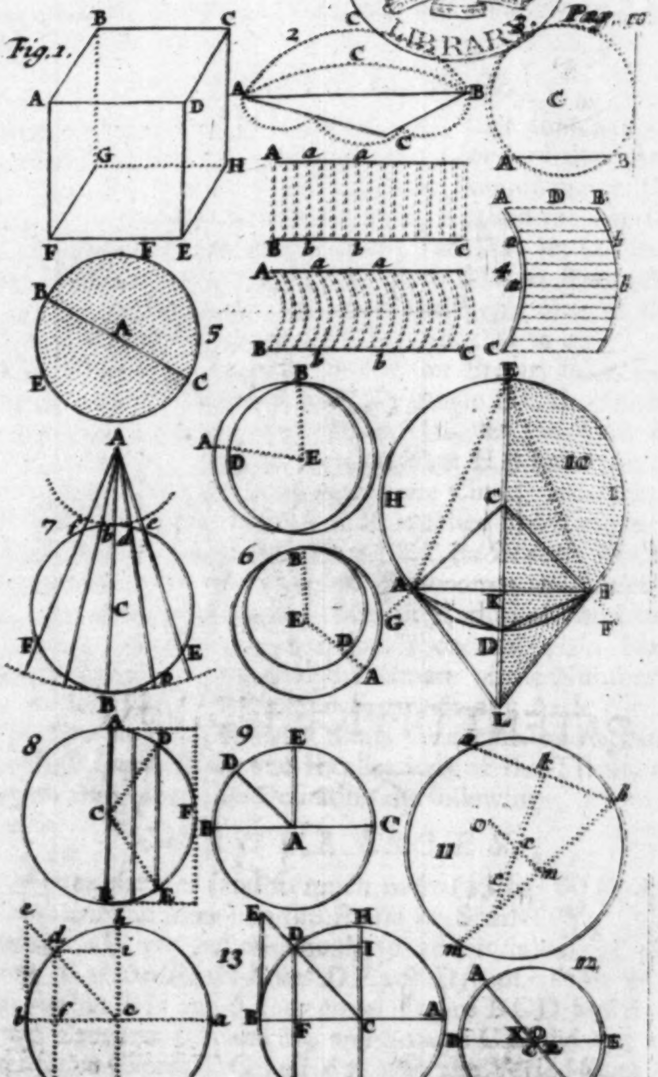

fig.	pag.
1.	5.
2.	6.
3.	7.
4.	ib.
5.	ib.
6.	ib.
7.	8.
8.	ib.
9.	ib.
10.	9.
11.	10.
12.	9.
13.	11.

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of X and O is equal (by Hypoth.) to the Sum of O and Z, it follows, that both the one and the other will make Semicircles, whoſe Diameters will be AB and DE, and ſo Right Lines.

DEFINITION X.

IN any Circle, a Right Line, as DG, that ſubtends any Arch, DBG thereof, is called the *Chord* of that Arch (*Fig.* 13.) BF (a Part cut off from the Semidiameter BC paſſing thro' the Middle of the Chord) is called the *Sagitta*, but moſt commonly the *Verſed Sine*; and DF let fall from the other Extremity of the given Arch BD, at Right Angles, on the Semidiameter, is called the *Right Sine* of that Arch BD, or of the Angle BCD; alſo DI is called the Right Sine of the Complement (or, for Brevity ſake, Sine Complement) of that Arch DH, or Angle DCH, &c. but the greateſt of all the Right Sines HC, let fall from the other Extremity (or End) of the Quadrant HB (which is indeed the ſame as the Semidiameter of the Circle) is called the *whole Sine* or *Radius*: laſtly, BE is called the *Tangent* of the Arch BD or Angle BCD, and CE its *Secant*. Whence Mathematicians, for the ſake of Trigonometrical Calculations, have divided the whole Sine or Radius of the Circle into 1000, 10000, 100000, 1000 000, 10000 0000, &c. Parts, thence to make a proportionable Eſtimate of the Number of Parts in the Sine, Tangent, or Secant of any Arch, &c. as may be ſeen in the Tables of Sines, Tangents, and Secants. From theſe Suppoſitions and Explications of the Terms, we ſhall now infer from this Definition the following

p. 10.

COROLLARIES.

1. IN equal Circles (and ſo much more (a) in one and the ſame) as the *Radii* or Semidiameters BC and *bc* are equal, ſo alſo it is evident, that the *Right Sines* DF and D*f*, of equal Arches BD and *bd*, or equal Angles BCD and *bcd*; alſo the Tangents BE and *be*, and Secants CE and *ce*, and Subtenſes or Chords DG and *dg*, alſo the Verſed Sines BF and *bf*, of the double Arches DBG and *dbg*, &c. will be equal, and ſo conſiſt of an equal Number of Parts of the whole Sine or Radius, &c. which both is evident from what we have ſaid before, and may be further evinced, if one Circle be conceived to be put on the other, and the Radius BC on the Radius *bc*, that ſo they may coincide, by reaſon of the

(a) *Among the reſt,* Vid. *Eucl.* 28 & 29. lib. 3. *and alſo partly the* 26, *and* 27.

the Equality of the Arches BD and *bd*; and ſo of all the reſt. And contrariwiſe,

II. In unequal Circles, the Sines, Tangents, *&c.* of equal Angles B*c*D or *bcd* (*Fig.* 14.) or ſimilar Arches; that is, Arches of an equal Number of Degrees, as BD and *bd*, will be alſo ſimilar or like, *&c. i. e.* the Sine *df* conſiſts of as many Parts of its *Radius bc*, as the Sine DF does of its Radius B*c*, *&c. e. g.* if the Radius B*c* be double of the Radius *bc*, each thouſandth part of the one, will be double of each thouſandth part of the other, but they are alike 1000 in each; becauſe the Degrees in the Circumference of the little Circle, particularly in the Arch *bd*, are but half as big as thoſe in the Arch BD, and yet equal in number in both. Thus alſo if the Sine DF contains 700 of the 1000 Parts of its Radius B*c*, *df* will alſo contain 700 of the 1000 Parts of its ſmaller Radius *bc*; and in like manner the Chords DG and *dg*, and the Tangents BE and *be*, *&c.* contain a like Number of Parts, each of its own Radius.

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SCHOLIUM.

IT may not be amiſs here to note by the by (altho' it may ſeem more proper to be taught after the Doctrine of Proportion) that if, *v. g.* the Degrees of a greater Circle be each of them reſpectively double, or triple, or quadruple, *&c.* of the Degrees of a leſs Circle, according as the Radius of the one is double or triple of the Radius of the other; then, at leaſt as far as Mechanical Practice can require, you may find an Arch of a greater Circle equal to the whole Periphery of a leſs, *viz.* If you take reciprocally ſuch a Part of the greater Periphery, which ſhall be as the Radius of the leſſer Circle, to the Radius of the greater; or as one Degree of the leſſer Periphery, to one Degree of the greater: *e. g.* if the leſſer Radius *bc* be half of the greater BC, and ſo alſo the Periphery, and each of the Degrees of the one, be one half of the Periphery, and of each of the Degrees of the other; then one half of the greater Periphery will reciprocally be equal to the whole leſſer Periphery, or 180 Degrees of the one, equal to 360 of the other, *&c.*

2. The ſame (at leaſt in this Caſe where the Radius *cb* is double of the Radius CB) may be done alſo geometrically by the ſame Reaſon. Having deſcribed Circles to each Radius, ſuppoſe the Radius CB (*Fig.* 15.) ſo to move with an equable Motion about its Center C, as to take or move the Radius of the greater Circle *cb* along with it, and coming,

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coming, *v. g.* to I. ſtops that alſo at 1, and going forward, to II. ſtops that again at 2, *&c.* Hence it will be manifeſt to any attentive Reader, that when the leſſer Radius CB ſhall have deſcribed the Semicircumference B. II. III. the greater Radius *c b* having moved to 3, will have deſcribed preciſely a quarter of its Circumference; and if ſtill the leſs Radius C. IV. moves on to the right hand, and continues to carry the greater *c* 4 along with it onwards the ſame way, it will neceſſarily follow, that in the ſame Moment as the Radius C. IV. (together with *c* 4.) ſhall come to its firſt Situation in B, having deſcribed a whole Circle, the oppoſite Radius *c* 4 will be come to 5, and have deſcribed half its Circle, having moved all along with an equable Motion. Hence it is evident, that the whole leſſer Circle anſwers exactly to half the greater, and half of the firſt to a quarter of the laſt; as alſo the Quadrant B. II. to the Octant (or 8th Part) *b* 2, &c. whence any Arch being given, as B. I. in the leſſer Circle, if you draw thro' I the Radius of the greater Circle *c* 1. you'll cut off an Arch *b* 1. equal to the given Arch in Magnitude, but only half in the Number of Degrees.

3. Hence follows naturally that celebrated Propoſition of *Euclid*, *viz.* that the Angle at the Center BC. I. or BC. II. is double of the correſponding Angle at the Periphery *b c* 1. or *b c* 2, *&c.* which in this Caſe is manifeſt, and in the other two (*Fig.* 16.) of the Wholes or Remainders DCD and DPD it is alſo (*a*) certain; becauſe by the firſt Caſe it is true of the Parts BCD and BPD to be added or ſubſtracted.

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(*a*) *Eucl. p.* 20. *l.* 3.

4. Hence we have a new way of biſecting any given Angle CDE, or Arch CE (*Fig.* 17.) *viz.* if you make CB equal to the Leg DC; and with this, as a Radius, deſcribe an Arch BF equal to the Arch CE, and draw DF.

Eucl. p. 9. *l.* 1.

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And with the ſame Facility we might obtain the Triſection of an Angle, if the greater Radius being triple to the leſs, was thus carried along by an equable Motion, as we have ſhewn how to do already in a double Radius; and this at firſt ſight may ſeem very probable.

But whether the triple Radius be immediately carried round by the ſimple Radius CB, (*Fig.* 18.) or by means of the double Radius *c b*, neither the one nor the other will cauſe an equable Motion. For in the latter Caſe, while the Radius *c b* deſcribes the Quadrant B*b*, the Radius *de* will

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will not deſcribe ſo much as a Quadrant; but while *c b*, with the ſame Velocity deſcribes the other Quadrant *b f*, the Radius *d e* will come to *g*; deſcribing an Arch as much greater than a Quadrant, as the former was leſs. In the former Caſe, on the contrary, the Radius CB would move on to D beyond a Quadrant, while *d e* was carried from B to *e*; but if the Radius CD moving on, ſhould again carry *d e* along with it, the one would deſcribe the ſame Arch *e* B back again, while the other would deſcribe one leſs than before.

p. 18.

5. Hence the Angle at the Center ACE (*Fig.* 19.) upon the Arch AE, is equal to the Angle ADB at the Periphery, upon double that Arch AB.

6. Hence the Angle ADB in a Semicircle (Num. 1.) (*a*) is a Right Angle, in a Segment leſs than a Semicircle (Num. 2.) is an obtuſe Angle, and in a greater (Num. 3.) an acute one; becauſe the Angle at the Center ACE upon the half Arch, is equal to the Angle ADB, *Pr.* 5. precedent, and is a Right Angle in the firſt Caſe, obtuſe in the ſecond, and acute in the third.

(*a*) *Eucl.* 31. *lib.* 3.

7. Hence Angles in the ſame Segment, or (*b*) on equal Segments of equal Circles, or on the ſame or equal Arches, are all equal and contrariwiſe.

(*b*) *Eucl.* 26, *and* 27. *lib.* 3.

DEFINITION XI.

WHEN two or more Lines AB and CD are ſo continued, as to keep always the ſame Diſtance from one another (whoſe Geneſis may be conceived to proceed from the uniform Motion of two Points A and C, always keeping the ſame Diſtance from each other) they are called *Parallels:* But as it evidently follows from this Definition, that (*a*) thoſe Lines which are parallel to ſome third, are parallel to one another (ſince adding or ſubſtracting equal Intervals to or from other equal ones, the Sums or Remainders muſt needs be equal) ſo if the Parallels are Right Lines, and cut acroſs by another Right Line EF, you'll have theſe

(*a*) *Eucl. lib.* 1. 30.

COROLLARIES.

I. THE Angles (*b*) which we call *alternate* ones, *viz.* GHK and HGI (*Fig.* 21.) are equal by *Corollary* I. *Definition* X. ſince the Diſtances

(*b*) *Eucl. lib.* 1. 29.

p. 18.
p. 11.

Diſtances GK and HI, which are the Right Sines of the ſaid Angles, are ſuppoſed equal.

II. The External Angle EGA is alſo equal to the Internal oppoſite Angle GHK, by *ConſeƐt.* 1. *Definit.* 9. becauſe that External Angle EGA is equal to the alternate Vertical one HGI. p. 10.

III. The ſame Internal Angle GHK, with the other internal oppoſite one on the ſame Side AGH (as well as the External one EGA, by *Coroll.* 1. *Definit.* 8.) are equal to two Right ones. p. 9.

IV. On the contrary; if any Right Line EF (*a*) cutting two others AB and CD acroſs, makes the alternate Angles GHK and HGI equal, their Right Sines, by *ConſeƐt.* 1. *Definit.* 10. will be equal, and conſequently, the Lines AB and CD parallel; and the ſame will follow, if the External Angle be ſuppoſed equal to the Internal, or the two Internal ones on the ſame Side, equal to two Right ones; ſince from either Hypotheſis, the former will immediately follow. p. 11.

(*a*) *Lib.* 1. *Prop.* 27, *and* 28.

V. From whence it appears, more than one way, (*b*) *That the three Internal Angles of any Triangle* (e. g. H, G, K, which will ſerve for all) *taken together, are equal to two Right ones, and the External one* GHD, *is equal to the two Internal oppoſite ones.* For we might either conclude with *Euclid*, that 1, 2, 3, together, make two Right ones, by *ConſeƐt.* 1. *Definit.* 8. but 2 = II and 3 = III *Pr.* 1 and 2 of this; therefore I, II, III = 2 Right ones; or with others, 1, II, 4 are = 2 R. but 1 = I and 4 = III *Pr.* 1. of this. Therefore, &c. or more briefly with *F. Pardies*, 1 = I *Pr.* 1. of this; but 1, II, III, together = to 2 Right ones, by the 3d of this. Therefore I, II, III = 2 R. Q. E. D. p. 9.

(*b*) *Lib.* 1. *Prop.* 32.

DEFINITION XII.

IF a Right Line AB (*Fig.* 22.) be conceived to move from the Vertex of a plain Angle CAD, with a Motion always parallel to it ſelf, ſo that at one End A, it ſhall always touch the Leg AC, and all along cut the Leg AD, while at length being come to F, it ſhall only touch that Leg with its other End B, and ſo fall wholly within the Angle CAD. By this Motion the Line AB will deſcribe within the Legs CAD, the Triangular Figure BAF; and without them, the Triangular Figure BAF: its Parts within them *af* continually increaſing, and the others without *fb* con- p. 18.

continually decreaſing; but with all its Parts, or the whole Line, it will deſcribe the Quadrangular Figure AEFB. Conſequently, if the other Leg AD of the given Angle CAD (*Fig.* 23.) or any Part of it AB, be moved along the other Leg remaining parallel to itſelf, it will alſo deſcribe a quadrilateral Figure, which will be alſo equilateral, if the Line deſcribing it AB, be equal to the Line AE according to which it is directed: But if either of the Lines, as AD, be greater than the other, the oppoſite Sides will be only equal; for the Deſcribent, or deſcribing Line, is always neceſſarily equal to it ſelf, and the Points A, B, D, moved with an equable Motion, deſcribe alſo in the ſame Time equal Lines AE, BF, DG. From theſe *Geneſes* of Quadrangles and Triangles, we have the following

p. 18.

CONSECTARIES.

I. THESE Quadrilateral Figures are alſo Parallelograms, *i. e.* they have their oppoſite Sides parallel; (*a*) becauſe the Line that deſcribes them, is ſuppoſed to remain always parallel to it ſelf, and the Points A and D, or A and B, to be always equidiſtant.

(*a*) *Schol. Prop.* 34. *lib.* 1.

II. Becauſe the two Internal oppoſite Angles (*b*) A and E, and alſo E and F, &c. are equal to two Right ones, by *Conſect.* 3. *Definit.* 11. if one Angle, *v. g.* that at A be a Right one, all the others muſt neceſſarily be ſo too, [in which caſe the quadrilateral and equilateral Figure AF is called a *Square*, and the other AG an *Oblong*, or *Rectangle*] If there be no Right Angle, the oppoſite Angles croſs-ways, *a* and *f*, or *a* and *g*, are equal, becauſe both the one and the other, with the third *e*, make two Right ones; [in which caſe the quadrilateral equilateral Figure *af*, is called a *Rhombus*, but the other *ag* a *Rhomboidis*.

p. 15.

(*b*) *The firſt Part of the ſame Prop.*

III. The Tranſverſal (or Diagonal) Line (*c*) as AF (in *Fig. preced.*) in any Parallelogram, divides it into two equal Triangles AEF and FAB; for all the Lines and Angles on each ſide are equal: and as the deſcribent (Line) AB moved thro' the Angle EAF upon the Line AE, deſcribed the Triangle AEF; ſo the Line EF, equal to the former, moved after the ſame way, thro' the Angle AFB, alſo equal to the former Angle, upon the equal Line FB, muſt neceſſarily deſcribe an equal Triangle; or, in ſhort, all the Indiviſibles

(*c*) *The latter Part of the ſame Prop.*

af, or their whole increaſing Series, are neceſſarily equal to the like Number of Indiviſibles fb, increaſing reciprocally after the ſame way.

IV. All Parallelograms that are between the ſame Parallels A b, and C f (*Fig.* 24.) *i.e.* having the ſame Altitude (*a*) and the ſame or equal Baſes, as CD or CD and cd, are equal among themſelves; for they may be conceived to be deſcribed by the equal Lines A B and ab equally moved thro' the ſame or equal Intervals of the Parallel Lines; ſo that all or each of the Indiviſibles or Elements A B will neceſſarily be equal to all and each of the Indiviſibles ab; for they all along anſwer one to the other both in Number and Magnitude.

(*a*) *lib.* 1. *Prop* 35 & 36.

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SCHOLIUM.

HERE you have a Specimen of the Method of Indiviſibles, introduced firſt by *Bonaventura Cavallerius*, and ſince much facilitated; and altho' theſe Indiviſibles, placed one by another, or as it were laid upon an Heap, cannot compoſe any Magnitude, yet by an imaginary Motion they may meaſure it, and, as it were, after a negative way, demonſtrate the Equality of two Magnitudes compared together, *viz.* if we conceive a certain Number of ſuch Elements in any given Magnitude, and thence conclude, that in another conſiſting of the like Elements, ordered or ranked after the ſame way, there can be neither more nor leſs in Number than in the firſt; thence follows their Equality, *&c.*

V. Hence therefore it is alſo evident, that Triangles upon the ſame or equal Baſes, as CD and cd, and placed between the ſame Parallels, are neceſſarily (*a*) equal, becauſe they are the halves of equal Parallelograms A D and ad, by the 3d *Conſectary* of this *Definition.*

(*a*) *Lib.* 1. *Prop.* 37 & 38, & 41.

p. 16.

VI. *F. Mourgues* ingeniouſly concludes from hence, *viz.* becauſe the two internal oppoſite Angles (*b*) on the ſame Side in any Parallelogram, are equal to two right ones, and ſo all together equal to four; that therefore the three Angles of any Triangle A B C (*Fig.* 25. which may always be compleated into a Parallelogram) are equal to half of thoſe four, or two right ones. This may be yet more briefly conceived thus; the Angles $b+c+d$ (the Sum of the two inferiour ones) $=$ to two right ones; but $a =$ alternate d: therefore $b+c+a =$ to two right ones. Q. E. D.

(*b*) *Lib.* 1. *Prop.* 32. *otherwiſe.*

p. 24

VII. Becauſe it is manifeſt in Rectangular Parallelograms, *p. 24.* if the Altitude A B, and Baſe B C (*Fig.* 26.) meaſured and divided by the ſame common Meaſure, be conceived to be (multiplied or) drawn one a-croſs the other, that the * Area A C, thereby deſcribed, will be divided into as many little ſquare Meaſures or Area's, as the Number of their Sides multiplied together would produce Units; therefore the Area of any other Parallelogram will be after the like manner produced, if the Baſe be multiplied by the perpendicular Height, equally as if it were a Rectangle of the ſame Baſe and Altitude.

** This is manifeſt from its Geneſis; for the five Parts of the Line A B by its Motion along the Part of the Baſe B E, deſcribes five little Squares, and moving along the following Part, deſcribes five more,* &c.

VIII. Conſequently alſo you may have the *p. 16,17.* Area of any Triangle, by the 3d and 5th *Conſectary*, if the Baſe be multiplied by half the perpendicular Height, or, the whole Baſe being multiplied by the Height, if you take the half of the Product.

DEFINITION XIII.

BUT as there are various Species of Triangles, while firſt, with relation to their Sides, one is called *Equilateral*, as *p. 24.* ABC (*Fig.* 27.) becauſe all its Sides are equal; another *Equicrural* or *Iſoſceles*, as D E F, becauſe it has two equal Sides D E and E F, while its Baſe D F may be either longer or ſhorter; and a third is called *Scalenous*, as G H I, becauſe it has all its Sides unequal: then again, in reſpect to their Angles, one is called *Rectangled*, as *a b c*, becauſe it has one right Angle at *a*; another *Obtuſe-angled*, as *d e f*, becauſe it has one obtuſe Angle at *d*; a third is called *Acute-angled*, as *g h i*, becauſe all its three Angles are acute: So each of theſe kinds has its peculiar Properties, which we ſhall partly hereafter demonſtrate in their proper Places, and partly deduce here, as

CONSECTARIES.

I. ALL Equilateral Triangles are alſo Equiangular, and conſequently Acute-angled; for having found a Center *p. 24.* for three Points, and the Periphery A B C, (ſee *Fig.* 28.) by *p. 9.* *Conſect.* 6. *Definit.* 8. the three Arches A B, B C, and A C, anſwering to equal Chords; and conſequently the three An- *p. 11.* gles at the Center O, are equal, by *Conſect.* 1. of *Definit.* 10. and therefore the three Angles at the Periphery alſo, as *p. 12.* being the Halves of the others, by the third *Conſectary* of the

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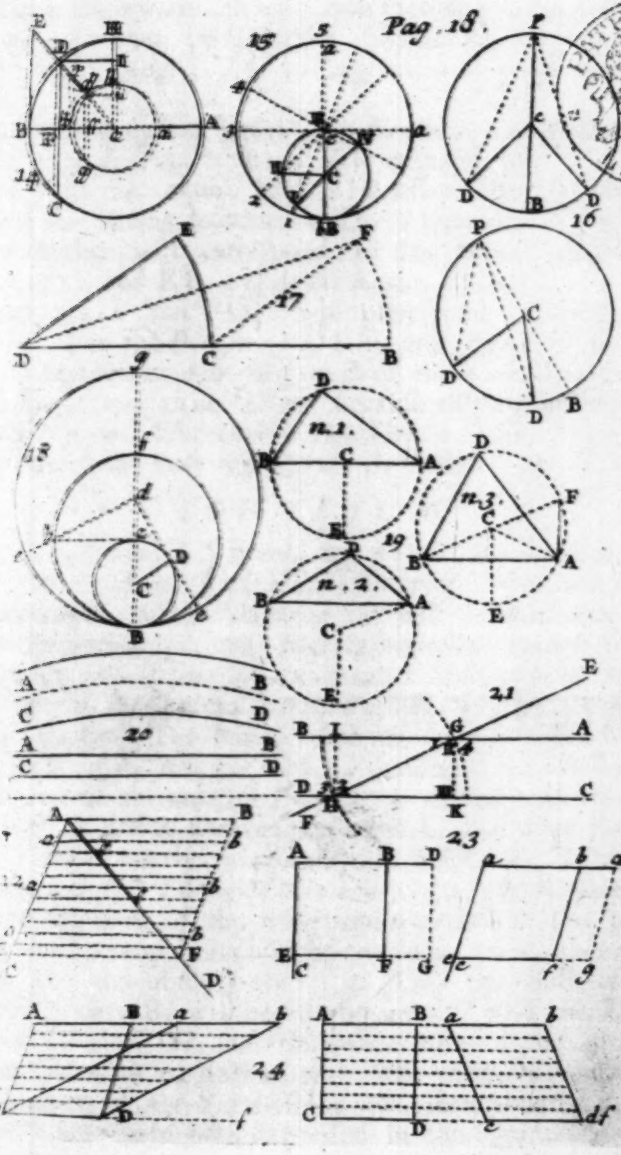

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the ſame *Definition.* Each Angle therefore is one third Part of two right ones, by *Conſect.* 6. *Definit.* 12. or two thirds of one right Angle, *i. e.* 60 Degrees, and conſequently Acute. p. 17.

II. It follows alſo, by the ſame Reaſon, in an *Iſoſceles* Triangle, that the Angles at the Baſe oppoſed to equal Sides are equal, and (*a*) conſequently acute; for having deſcribed a Circle about it, equal Arches will correſpond to the equal Chords D E and E F, and equal Angles at the Center D O E and F O E will correſpond to them, and equal ones at the Periphery D F E, and F D E to theſe again. And it is evident that each of theſe are leſs than a right Angle, *i. e.* an acute one, becauſe all three are equal to two right ones. Wherefore if the third is a right Angle, the other two at the Baſe will neceſſarily be half right ones.

(*a*) *Lib.* 1. *Prop.* 5. *the ſame otherwiſe demonſtrated.*

SCHOLIUM.

WE will here (*a*) ſhew, by way of Anticipation, the Truth of the *Pythagorick Theorem*, eſteemed worth an *Hecatomb:* which hereafter we will demonſtrate after other different ways; *viz. In a right-angled Triangle B A C,* (*Fig.* 29.) *the Square of the greateſt Side, oppoſite to the right Angle, is equal to the Squares of the other two Sides taken together.* For having deſcribed the Squares of the other two Sides, A C *d* E, D E *a b* (taking E D = A B) and the Square of the greateſt B C *c b*, it will be evident, that the Parts X and Z are common to each, and that the two other Triangles in the greateſt Square B A C and B D *b*, are equal to the two Triangles *b a c* and C *d c*, which remain in the two Squares of the leſſer Sides; and ſo the whole Truth of the Propoſition will be evident, while theſe two things are undoubtedly true: 1. That the Side of the greateſt Square B *b* will neceſſarily concur with the Extremity of the leſs D *b*, and the other Side of the greateſt Square C *c* with its Extremity *c*, will preciſely touch the Continuation of the Sides of the two leaſt Squares *d* E *a*; as you'll ſee them both expreſſed in the Figure. 2. The ſaid two Triangles are every way equal; for the Angles at C, with the intermediate one at Z, make two right ones, therefore they are equal; but the Side C A is equal to the Side C *d*, and C B to C *c*, and the Angles at A and *d* right ones. Wherefore if we conceive the Triangle A B C to be turned about C, as a Center, to the right hand, it p. 24.

will exactly agree with the Triangle C*dc*, and the Point B will neceſſarily fall on the continued Line *d*E, as agreeing with the Line AB. Hence it is now evident, that C*a* = BD, and becauſe *ba* is alſo = *b*D, and the Angles at *a* and D right ones; therefore if we conceive the Triangle *bac* to be moved about *b* as a Center, until *ba* coincides with *b*D, and *ac* with DB, *bc* will alſo neceſſarily coincide with *b*B. *Q. E. D.*

To this Demonſtration of *Van Schooten*'s, which we have thus illuſtrated and abbreviated, we will add another of our own, more like *Euclid*'s, but ſomewhat eaſier, which is this: Having drawn the Lines (as the other Figure the 29th directs) the △ ACD being on the ſame Baſe AC with the Square AI, and between the ſame Parallels, is neceſſarily one half of it, but it is alſo half of the Parallelogram CF, being on the ſame Baſe with it, *viz.* DC; therefore this Parallelogram = □ AI. In like manner △ ABE is half the □ AL, and alſo half the Parallelogram BF, therefore BF = □ AL: therefore CF + BF, that is the □ of BD = to the two □ AI + AL. *Q. E. D.* For that the Side BE occurs to, or meets the Side LK, and the Side CD the Side IH continued, it apparently follows from hence, *viz.* becauſe the Angles *a* and *b*, and alſo *c* and *d*, are manifeſtly equal, as making right Angles both ways, with the intermediate one X or Z. Therefore the △ BAC being turned on the Center B, will exactly agree with BLE, and turned on the Center C, will agree with CID alſo, *&c.*

DEFINITION XIV.

ALL rectilinear or right-lined Figures, that have more than three or four Sides (to the latter ſort of which there remains to be added another Species beſides Parallelograms, called *Trapeziums*, whoſe Angles and Sides are un-
30 equal, as KLMN, *Fig.* 1c.) are called by one common Name *Polygons*, or *Many-ſided* and *Many-angled Figures*, and particularly according to the Number of their Sides and Angles, *Pentagons*, *Hexagons*, *Heptagons*, &c. All whereof, as alſo *Trapezia*, being reſolvable into Triangles by Diagonal Lines (as may be ſeen in the 31ſt and foregoing Fig.) you have theſe

CON-

CONSECTARIES.

I. YOU have the Area of any Polygon by reſolving it into Triangles, and then adding the Area's of each Triangle found by *Conſect.* of *Definition* 12. into one Sum. p. 16.

II. The Area of the *Trapezium* KLMN (in Numb. 1. *Fig.* 30.) whoſe two oppoſite Sides, at leaſt KL and MN, p. 24. are Parallel, may be had more compendiouſly, if the Sum of theſe Sides be multiplied by half the common height, KO.

SHOLIUM.

HENCE we have the Foundation of *Epipedometry* or meaſuring of Figures that ſtand on the ſame Baſe, and Ichnography; in the practice whereof this deſerves to be taken ſpecial notice of, that to work the more compendiouſly, you ought to divide your Figure into Triangles, ſo that (*Fig.* 31.) two of their Perpendiculars may (as conveniently as p. 24. can be) fall on one and the ſame Baſe. For thus you'll have but one Baſe to meaſure, and two Perpendiculars to find the Area of both: but for Ichnography, the diſtance of the Perpendiculars from the neareſt End of the Baſe muſt be taken; which we ſhall ſuperſede in this place, and diſcourſe more largely on hereafter.

II. This Reſolution of a Polygon into Triangles may be performed by aſſuming a Point any where about the Middle, and making the Sides of the Polygon the Baſes of ſo many Triangles; (ſee the 2d Figure mark'd 31) wherein it is evident, 1. That all the Angles of any Polygon are equal to twice ſo many right ones, excepting four, as the Polygon has Sides; for it will be reſolved into as many Triangles as it has Sides, and each of theſe has its Angles equal to two right ones. Subſtracting therefore all the Angles about the Point M (which always make four right ones by *Conſ.* 2. *Def.* 8.) p. 9. there remain the reſt which make the Angles of the Polygon. 2. All the external Angles of any right-lined Figure (*e*, *e*, *e*, &c.) are always equal to four right ones; for any one of them with its contiguous internal Angle is equal to two right ones *per Conſect.* 1. of the ſaid *Def.* and ſo altogether equal p. 9. to twice as many right ones as there are Sides or internal Angles of the Figure. But all the internal ones make alſo twice ſo many right ones, excepting four, therefore the external ones make thoſe four.

DEFINITION XV.

AMONG all theſe plain Figures, thoſe are call'd *Regular*, whoſe Angles and Sides are all equal; as among trilateral Figures, the equilateral Triangle; among quadrilateral ones, the Square; and in other kinds, ſeveral Species which are not particulariz'd by Names; but all others, in whoſe Angles or Sides there is any Inequality, are called *Irregular*: Tho' ſome of theſe alſo, and all the others may be inſcribed in a Circle. Whence you have theſe

CONSECTARIES.

I. THE Area's of the regular Figures may be obtained
p.9,10. yet eaſier, if having found their Center (by *Conſect.* 6.
Definit. 8.) you draw from thence the right Lines CB, CA,
p.24. &c. (*Fig.* 32.) till there be form'd as many Triangles ACB,
BCF, &c. as the Figure has Sides; for ſince all theſe Triangles
have their Baſes AB, BF, as ſo many Chords, and
their Altitudes CD, CG, as ſo many Radii leſſened by the
verſed Sines DE and GH, which are all equal *per Conſect.*
p.12,14. 1, 5. *Definit.* 10. one of their Area's being found and multiplied
by the Number of Sides, or half the Altitude by the
Sum of all the Sides, and you'll have the Area of the whole
Polygon: For it is manifeſt from what we have already ſaid,
and very elegantly demonſtrated by *F. Pardies*, *That any
regular Polygon inſcribed in or circumſcribed about a Circle,
is equal to the Triangle A*za, *one Leg whereof is equal to
the perpendicular Height let fall from the Center upon any
Side, and the other to the whole Periphery of the Polygon.*
Now if the Triangles, into which the Polygon is reſolved, do
p.24. all ſtand on the ſame right Line A*a*, (*Fig.* 32.) and are all
equal and of the ſame height, to which the Perpendicular
AZ is equal, it will neceſſarily follow, that each Pair of Triangles
ABZ and ABC, BZF and BCF, &c. are equal among
p.17. themſelves, *per Conſect.* 5. *Definit.* 12. and conſequently
the Sum of all the former will be equal to the Sum of all
the latter, that is, the Triangle A*za* to the Polygon given.

II. Since regular Figures inſcribed in a Circle, by biſecting their Arches AB, BF, &c. may be eaſily conceived to be changed into others of double the number of Sides, (as a Pentagon into a Decagon, &c.) and that *ad Infinitum*; a Circle may be juſtly eſteem'd a Polygon of infinite Sides, or conſiſting of an infinite Number of equal Triangles, whoſe common Altitude is the Semidiameter of the Circle: So that the Area of any Circle is equal to a right-angled Triangle (as A Z *a*) one of whoſe Sides A Z is equal to its (*a*) Semidiameter, and the other A *a* to its whole Circumference.

(a) Archimedes of the Dimenſion of the Circle, Prop. 1.

SCHOLIUM.

IT may not be amiſs to note theſe few Things here, concerning the Inſcription of regular Figures in a Circle.

I. Having deſcribed a Circle on any Semidiameter A C, (*a*) (*Fig.* 33. *N.* 1.) that Semidiameter being placed in the Circumference, will preciſely cut off one ſixth part of it, and ſo become the Side of a regular Hexagon: and ſo the Triangle A B C will be an equilateral one, and conſequently the Angle A C B and the Arch A B 60 Degrees, by *Conſ.* 1. *Definit.* 13.

(a) Eucl. Prop. 15. lib. 4. p. 26.

p. 18.

II. Hence a right Line A D, omitting one Point of the Diviſion B, and drawn (*b*) to the next D, gives you the Side of a regular Triangle inſcribed in the Circle, and ſubtends twice 60, *i. e.* 120 Degrees.

(b) Eucl. 2d Coroll. of the ſame.

III. If the Diameters of the Circle A D and B E (*N.* 2.) cut one another at right Angles in the Center C, the right Lines A B, B D, *&c.* will be the Sides of an inſcribed Square A B D E: For (*d*) the Sides A B, B D, *&c.* are the equal Chords of Quadrants, or Quadrantal Arches, and the Angles A B D, B D E, *&c.* will be all right ones, as being Angles in a Semicircle (*per Schol.* 6. *Definit.* 10.) compoſed each of two half right ones, by *Conſect.* 2. *Definit.* 13.

(d) Eucl. 6. lib. 4.

p. 14.

p. 19.

IV. *Euclid* very ingeniouſly ſhews us how to inſcribe a regular Pentagon alſo, *Lib.* 4. *Prop.* 10 & 11. and alſo a Quindecagon (or Polygon of 15 Sides) *Prop.* 16. But though the firſt is too far fetch'd to be ſhewn here, yet (ſuppoſing that) the ſecond will eaſily and briefly follow:

In a given Circle from the ſame Point A (*N.* 3.) inſcribe a regular Pentagon A E F G H A, and alſo a regular Triangle A B C; then will B F be the Side of the *Quindecagon*, or 15 ſided Figure. For the two Arches A E and E F make together 144 Degrees, and A B 120: Therefore the difference B F will be 24 Deg. which is the 15th Part of the Circumference.

V. * The Invention of *Renaldinus* would be very happy, if it could be rightly demonſtrated; (as he ſuppoſes it to be in his Book of the Circle) which gives an univerſal Rule of dividing the Periphery of a Circle into any number of equal Parts required, in his 2d Book *De Reſol. & Comp. Mathem. p.* 367. which, in ſhort, is this: Upon A B the Diameter of a given Circle (*Fig.* 34.) make an equilateral Triangle A B D, and having divided the Diameter A B into as many equal Parts, as you deſign there ſhall be Sides of the Polygon to be inſcribed, and omitting two, (*e. g.* from B to A) draw a right Line thro' the beginning of the third from D, to the oppoſite concave Circumference, and thence another right Line to B the end of the Diameter, which the two Parts you omitted do touch. So, *e.g.* for the Triangle, having divided A B into three equal Parts, if omitting the two B 2, thro' this beginning of the third you draw the right Line DIII, and thence the right Line IIIB, this will be the Side of the Triangle; and ſo IV.B will be the Side of the Square, V.B the Side of the Pentagon, *&c.*

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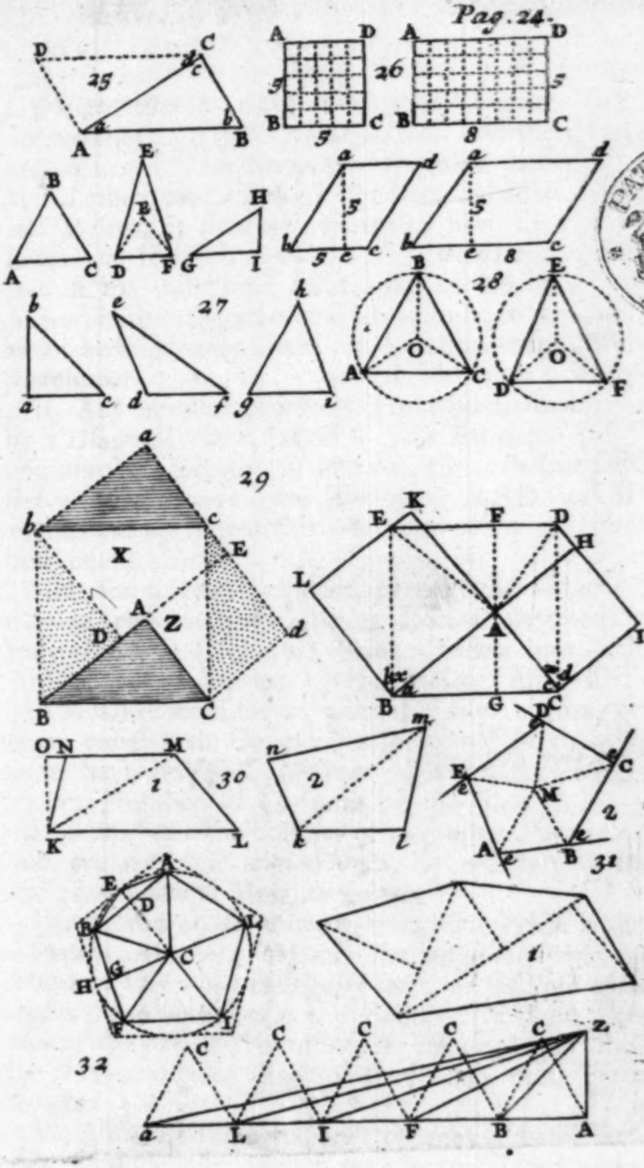
* *This Method is falſe, as it is eaſy to demonſtrate trigonometrically.*

N. B. The *Demonſtration* of theſe (which *Renaldinus* adds, *p.* 368.) we have ſeveral ways proſecuted in our Treatiſe of the Circle: Some of the moſt noted ancient Geometricians have ſpent a great deal of Pains in the Inveſtigation and Effection of this Problem, and ſeveral of the Moderns have loſt both Time and Pains therein: Whence, we hope, without the Imputation of Vain-glory, we may have ſomewhat obliged Poſterity in this Point.

DEFINITION XVI.

p. 26. IF the Plane of any Parallelogram A C (*Fig.* 35.) be conceived to move along a right Line A E, or another Plane A F downwards, and remaining always parallel to itſelf; there will be generated after this way a Solid, having ſix oppoſite Planes parallel, two whereof, at leaſt, will be equal to one another, whence it is called a *Parallelepipedon*; and particularly a *Cube* or *Hexahedron*, if the *Parallelogram* A B C D that deſcribes it be a Square, and the Line A E along which it is moved, equal to the Side of that Square, and perpendicular to the deſcribing Plane, and conſequently all the ſix parallel Planes comprehending this Solid, equal to one another. p. 26. But if the deſcribing Plane (*Fig.* 36.) be a Triangle or Polygon, the Solid is called a *Priſm*; if a Circle, it is called a *Cylinder*. Now from the Geneſis of theſe Solids, you have the following



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CONSECTARIES.

I. IF the Planes or deſcribing Parallelograms (*a*) ABCD and *abcd* (*Fig.* 37.) are equal, and their Lines of Motion AE and *ae* alſo equal; the Solids thereby deſcribed, *viz.* Parallelepipedons, Cylinders, and Priſms, (which will therefore have their Baſes and Heights equal) will be equal among themſelves; becauſe the deſcribent Indiviſibles of the one, will exactly anſwer, both in Number and Magnitude, to thoſe of the other, as we have already ſhewn in Parallelograms; conſequently therefore,

p. 26.

(*a*) Eucl. l. 11. p. 29, 30, 31.

II. Any Parallelepipedon (*b*) may be divided by a Diagonal Plane BDHF (or a Plane paſſing thro' its Diagonals) into two equal Priſms; for by *Conſect.* 3. *Definit.* 12. the Triangles ABD and BCD, are equal, and are ſuppoſed to be moved by an equal Motion thro' equal Spaces.

(*b*) Eucl. l. 11. p. 28.

p. 16.

III. And ſince it is evident, even by this Geneſis of them, that in right-angled Cubes and Parallelepipedons, if the Baſe ABCD (*Fig.* 38.) being divided into little ſquare Area's, be multiplied by the Height AE, divided by a like Meaſure for Length, after this way you may conceive as many equal little Cubes to be generated in the whole Solid as is the Number of the little Area's of the Baſe multiplied by the Number of Diviſions of the Side AE; you may moreover obtain the Solidity of any other Parallelepipedons, that are not right-angled ones, by multiplying their Baſes and perpendicular Heights together.

p. 26.

IV. Moreover, ſince every triangular Priſm is the half of a Parallelepipedon, and any multangular Priſm may be reſolved into as many triangular ones, as its Baſe contains Triangles; you may obtain the Solidity (or ſolid Content) either of the one or the other, if you multiply the triangular or multangular Baſe of them into their perpendicular Height.

V. After the ſame manner you may likewiſe have the Solidity of a Cylinder, which may be conſidered as an infinite-angled Priſm, juſt as the Circle is as an infinite-angled Polygon.

DEFINITION XVII.

IF ſome Triangle ABC (*Fig.* 39. *N.* 1.) be conceived to move with one of its Plane Angles A, from the Vertex or Top of any ſolid Angle (determined by two Planes *aAb* and

p. 26.

cAa

*c*A*a* join'd together in the common Line A*a*) with a Motion always parallel to itſelf; ſo that its extreme angular Point A ſhall always remain in the Line A*a*, and its Sides AB and AC ſhall all along touch or raze along the two angular Planes *a*A*b*, *c*A*a*, till at length it falls wholly within the ſolid Angle at *abc*: By this its Motion it will deſcribe within the ſolid Angle, a Figure we call *Pyramidal*, whoſe Baſe will be the Triangle *abc*, and its Vertex A. And it will alſo deſcribe, without the ſolid Angle, another quadrangular Pyramid, whoſe common Vertex will be the ſame Point A, but the Baſe the Quadrangle C*b*, deſcribed by the Side B*c* of the moveable Triangle: The firſt Pyramid it will deſcribe with its triangular Parts $\alpha\beta\gamma$ continually increaſing from the Point A, and ending in the Triangle *abc*; but the latter Pyramid will be deſcribed by the remaining Parts, continually decreaſing downwards from the whole ABC, and the quadrangular Trapezia $\beta\gamma\delta\varepsilon$ at length ending in the right Line *bc*: So that in the mean while the ſaid Triangle with its whole Space deſcribes, according to what we have ſaid before, the triangular Priſm compoſed of thoſe two Pyramids. From this Geneſis of Pyramids, you'll have the following

CONSECTARIES.

I. LET the deſcribing Triangles ABC, 𝔄𝔅ℭ (*N.* 2.) be what they will, ſo they are equal; and whatever the ſolid Angles comprehended under the Planes *ab*A, and *ac*A, 𝔞𝔟𝔄 and 𝔞𝔠𝔄 are, ſo they are accommodated to the Plane Angles A and 𝔄, and ſuch that the deſcribing Triangles *abc*, 𝔞𝔟𝔠 moved thro' equal Spaces from the Line AC to *ac*, and from the Line 𝔄𝔠 to 𝔞𝔠,may intirely fall within them at the ſame time; becauſe the Elements continually increaſing on both ſides, are neceſſarily conceived to be equal both in Number and Magnitude, and the ultimate ones *abc*, 𝔞𝔟𝔠, are equal by the Hypotheſis; therefore the Sums of them all on both ſides are equal, and ſo the equal triangular Pyramids *abc*A, and 𝔞𝔟𝔠𝔄, are deſcribed within thoſe ſolid Angles, and the equal quadrangular ones *bc*CBA, 𝔟𝔠ℭ𝔅𝔄 without them, and the equal Priſms A*abc*C, 𝔄𝔞𝔟𝔠ℭ; whence Pyramids of equal Baſes and Altitudes may be indoubtedly taken as equal to one another.

II. By cutting a Priſm a-croſs, or diagonally, (*a*) you do not divide it into two equal Parts, as in Parallelograms, but you cut it into aquadrangular Pyramid *cb*BCA (*Fig.* 40.) and into a triangular

Pyramid

(*a*) *Prop.* 7. *lib.* 12. *Eucl.*

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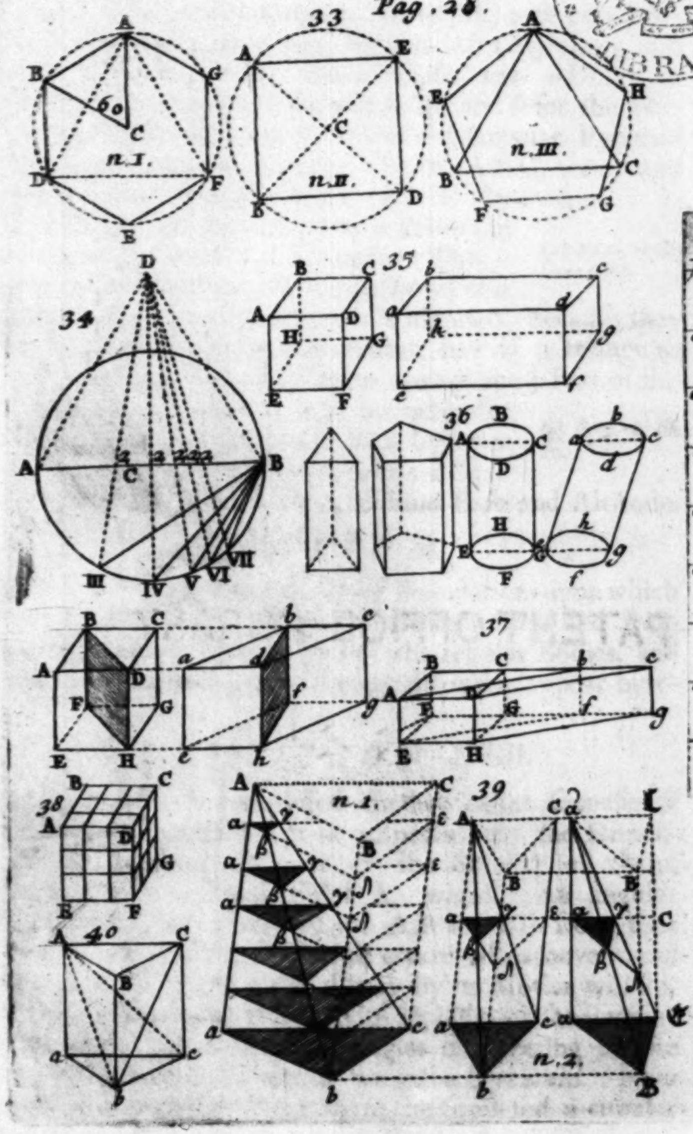

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Pyramid *abc*A, whereof the former is the double of the latter; for drawing the Diagonal *b*C, the Priſm may be conceived to be divided into two triangular Pyramids *b*BCA and *bc*CA, having the common Vertex A, and equal triangular Baſes, which, therefore, by the laſt *Conſectary*, are equal to one another. But one of theſe, *viz.* *b*BCA or ABC*b*, if ABC be taken for the Baſe, and *b* for the Vertex, is manifeſtly equal to the former triangular Pyramid *abc*A, becauſe they have equal Baſes ABC, *abc*, and the ſame common Altitude A*a* or *b*B of the Priſm.

III. Therefore a Pyramid is $\frac{1}{3}$ of a Priſm (*a*) having the ſame Baſe and Altitude. This is evident already (from the laſt *Conſectary*) of a triangular Priſm. And of others it is manifeſt, becauſe they may be divided into triangular Priſms, having a triangular Pyramid anſwering to each of them, being the $\frac{1}{3}$ Part of it.

(*a*) *Schol. to the ſame Prop.*

IV. And ſince a Cone (*b*) may be taken for a Pyramid of infinite Sides, and alſo a Cylinder for an infinite angled Priſm: therefore a Cone ſhall be $\frac{1}{3}$ of a Cylinder, having the ſame Baſe and Altitude.

(*b*) *Prop.* 10. *lib.* 12.

SCHOLIUM.

HENCE we now have the chief Foundations upon which is built the whole Buſineſs of Stereometry, and from which we may get the Dimenſions of the regular Bodies, and of Globes and Spheres; as will more diſtinctly appear hereafter.

DEFINITION XVIII.

THERE may be exhibited another eaſier Geneſis of Cones and Pyramids, but it reſpects only the Dimenſion of the Surface, and not of the Solidity of them, *viz.* If you take a fixed Point A, without an angular Plane (*Fig.* 41.) and a right Line AF let fall from that Point to any Angle of the Plane, be conceived to move round the Sides BC, CD, *&c.* this Plane by its Motion will deſcribe as many Triangles BAC, CAD, *&c.* as the angular Plane has Sides. And theſe Triangles all meeting at the Point A, make that Solid which we call a *Pyramid.* Now if inſtead of an angular Plane there be ſuppoſed a circular one, (or an angular one of infinite Sides) the Solid thence produced is called a *Cone*, whoſe Surface is equal to infinite Triangles, conſtituted on the Baſe BCDE, and whoſe Solidity would conſequently be equal to an infinite-angled Pyramid

p. 34.

ramid of the ſame Height. And after the ſame manner, by the Motion of the Line AF being always parallel to itſelf about Parellelograms, triangular Planes, and multangular Planes, will be generated Parallelepipedons, Priſms, and Cylinders. But as one Pyramid will be produced more upright than another, according as the Point A ſtands more
p.34. over the middle of the Plane BCD, &c. (*Fig.* 42.) or reſpects it more obliquely: So in particular a *Cone* is called a *right Cone*, when the Line AO being let fall to the Center of the circular Plane (otherwiſe called the Axis of the Cone) conſtitutes on all Sides right Angles with it; but it is called an *Oblique* or *Scalene* Cone, when the Axis ſtands obliquely on the Baſe. Which Diſtinction may alſo be eaſily underſtood when applied to Cylinders, tho' a right Cone and Cylinder may alſo be conceived to be generated after another way, as if for the one a right-angled Triangle AOB, and for the other a right-angled Parallelogram LAOB be conceived to be moved round a Line AO conſidered as immovable (whence it is called an Axis) and alſo a truncated Cone may be formed if a right-angled Trapezium, two of whoſe Sides are parallel, be moved, &c. And as we have deduced the Solidity of theſe Bodies, from the foregoing Geneſis, ſo their external Surfaces, as alſo thoſe of Priſms and Parallelepipedons, may eaſily be found from the preſent Geneſis, by any one who attentively conſiders the following

COROLLARIES.

I. SINCE the whole external Surface of any Pyramid, except the Baſe, is nothing but a Syſtem of as many Triangles ABC, CAD, &c. as the Pyramid has Sides; if
p.18. the Area's of thoſe Triangles ſeparately found by *Conſect.* 8. *Defin.* 12. be added into one Sum, you'll have the ſuperficial Area of the whole Pyramid.

II. If a Pyramid be cut with a Plane *bcde*, parallel to
p.34. its Baſe BCDE (*Fig.* 41.) the Surface of that truncated Pyramid, comprehended between the parallel Planes, may be obtained, if having found the Surface of the Pyramid A*bcde* cut off from the reſt by *Conſect.* 1. you ſubſtract it from the Surface of the whole Pyramid.

III. The external Surface (a) of a right Pyramid, that ſtands on a regular polygoneous Baſe is equal to a Triangle, whoſe Altitude is

(a) *Archimedes, lib.* 1. *Prop.* 7, & 8. *de Con.* & *Cylin.*

equal to the Altitude of one of the Triangles which compoſe it, and its Baſe to the whole Circumference of the Baſe of the Pyramid.

IV. Therefore the Surface of a right Cone, by what we have already ſaid, is equal to a Triangle, whoſe Height is the Side of the Cone, and the Baſe equal to the Circumference of the Baſe of the Cone.

V. The Surface of a truncated right Cone, or Pyramid, is equal to a Trapezium, which has two parallel Sides, the lower of which is equal to the Periphery of the Baſe, and the other to the Periphery of the Top, or upper Part, and the Height to the intercepted Part of the Side, or Altitude; ſo that the ſuperficial Area, in both Caſes, is eaſily had by *Cor.* 2. *Defin.* XIV. p. 21.

VI. The Surface of a right Cylinder, or Priſm, is equal to a Parallelogram, which has the ſame Height with them, and for its Baſe a right Line equal to the Periphery of that Cylinder or Priſm.

DEFINITION XIX.

IF a ſemicircular Plane ADB (*Fig.* 43. *N.* 1.) be conceived to move round its Diameter A B, which is fixed as an Axis; by this Motion it will deſcribe a *Sphere*, and with its Semicircumference the Surface of that *Sphere*; every Part whereof is equally diſtant from C, the middle Point of that Axis, which is therefore called the Center of that *Sphere*. Now if (*N.* 2.) the Periphery of this Semicircle (*a*) be conceived to be divided before that Revolution, firſt into two Quadrants A D, and B D, and then each of thoſe again into as many Parts equal in Number and Magnitude as you pleaſe; and having drawn the Chords A F, F E, E D, &c. let the Polygon A F E D G H B, inſcribed in the Semicircle, be conceived, together with it, to be turned about the Axis A B; then will the Triangles A 1 F, and B 4 H deſcribe two Cones about the Diameters F *f*, and H *h*; and the Trapezia about the Axes 1, 2, 2 C, C 3, and 3, 4, will deſcribe ſo many truncated Cones, and the Lines A F, F E, &c. ſo many conical Superficies, by the foregoing *Definition*; and ſo the whole polygoneous Plane A F E D G H B a conical Body inſcribed in the Sphere, and contained only under conical Surfaces. And as any attentive Perſon may eaſily perceive ſuch a Body to be leſs than the ambient Sphere, and its whole Surface leſs than the Surface p. 34.

(*a*) *Archimedes, lib.* 1. *de Cono & Cyl. Prop.* 22. *Coroll. & Prop.* 23.

of the ambient Sphere ; ſo he may as eaſily trace theſe following

CONSECTARIES.

I. IF the Arches A F, F E, &c. be further biſected, and a Polygoneous Figure of double the Number of Sides inſcribed in the Semicircle, and conceived to be moved round after the way we have already ſhewn, the Pſeudoconical Body hence ariſing, will approach nearer and nearer to the Solidity of the Sphere, and the Surface of the one to the Surface of the other ; and hence (if we continue this Biſection, or conceive it to be continued *ad infinitum*) you may infer

II. That a Sphere may be looked upon as ſuch a Pſeudoconical Body, conſiſting of infinite Sides, and its Surface will be equal to the infinite Conical Surfaces of that Body, which we will take further Notice of hereafter.

DEFINITION XX.

p. 34.

IF the Diameter AB of the Semicircular Plane A D B, (*Fig.* 43. *N.* 3.) be conceived to be divided into equal Parts (as here the Semidiameter A C into 3.) and if the circumſcribing Parallelograms C E, 2 F, 1 G, on the tranſverſe Parallels C D, 2 *e*, 1 *f*, be conceived together with the Semicircle itſelf to revolve about the fixed Axis A B ; it is evident that there will be formed a Sphere from the Semicircle as before, and from the circumſcribed Parallelograms ſo many circumſcribed Cylinders of equal Heights : but if all the Altitudes or Heights of theſe are biſected or divided into two equal Parts, and ſo make the Number of circumſcribing Parallelograms double, there will be formed (by moving them round as before) double the Number of Cylinders of half the Height ; but which yet, being taken together, approach much nearer to the Solidity and Roundneſs of the Sphere than the former, which were fewer in Number (*viz.* the ſix latter Parallelograms approach nearer to the Plane of the Circle than the three former :) and thus if that Biſection of the Altitudes be conceived to be continued *ad infinitum*, the innumerable Number of thoſe infinitely little Cylinders will coincide with the Sphere itſelf. Moreover, if you conceive any polyhedrous or multilateral Figure, to be circumſcribed about the Sphere, (which we here endeavour to delineate by the Polygon A B C D, &c. *N.* 4. circumſcribed about the Circle) and the ſolid Angles thereof to be cut by other Planes *a b*, which ſhall touch the Sphere ; it is manifeſt

feſt

ſect there will thence ariſe another polyhedrous Figure, the Solidity whereof will approach nearer to the Solidity of the Sphere, and its Surface to the Spherical Surface, than the former: and if the Angles of this be again in like manner cut off, there will ſtill ariſe another new Solid and new Surface, approaching yet nearer to the Solidity and Surface of the Sphere than the former, *&c.* and ſo, after an infinite Proceſs, they will coincide with the Sphere and its Surface themſelves. Whence flow theſe

COROLLARIES.

I. A Sphere may be conſidered as a Polyhedrous Figure, conſiſting of innumerable Baſes, *i.e.* compoſed of an innumerable Number of Pyramids, having all their Vertexes meeting in the Center, and ſo their common Height is the Semidiameter of the Sphere, and the Sum of all their Baſes is equal to the Superficies of the Sphere.

II. If you can find the Proportion between a Cylinder of the ſame Height with any Sphere, having its Baſe equal to the greateſt Circle of that Sphere, and between innumerable Cylinders circumſcribed about it, as we have juſt now ſhewn; then you may alſo obtain the Proportion between the ſaid circumſcribed Cylinder and the inſcribed Sphere: Which to have here hinted, may be of ſervice hereafter in its proper place.

DEFINITION XXI.

THERE remain to be conſidered thoſe Bodies which are called *Regular*, which correſpond to the Regular Plane Figures; and as thoſe conſiſt of equal Lines and Angles, ſo theſe likewiſe are comprehended under regular and equal Planes meeting in equal ſolid Angles; and as thoſe may be inſcribed and circumſcribed about a Circle, ſo may the the latter likewiſe be deſcribed in and about a Sphere. But whereas there are infinite Species of Regular Plane Figures, yet there are only five of the Regular Solids; the firſt whereof is contained under four equal and equilateral Triangles, whence it is named a *Tetrahedron*; the ſecond is terminated by ſix equal Squares, and thence it is called a *Hexahedron*, and otherwiſe a *Cube*; the third being comprehended under eight equal and equilateral Triangles, is called an *Octahedron*; the fourth is contained under twelve regular and equal Pentagons, and ſo is named a *Dodecahedron*; Laſtly, the fifth is contained under twenty regular and equal Triangles,

gles, and is thence nominated an *Icoſahedron.* Beſides theſe five ſorts of regular Bodies, there can be no others; for from the Concurrence of three equilateral Triangles ariſes the Solid Angle of a *Tetrahedron*; from four, the Solid Angle of an *Octahedron*; from five, the Solid Angle of an *Icoſahedron*; from the Concurrence of four Squares, you have the Solid Angle of an *Hexahedron*; from that of three Pentagons you have the Solid Angle of a *Dodecahedron:* and in all this Collection of Plane Angles, the Sum does not ariſe ſo high as to four right ones. But four Squares, or three Hexagons meeting in one Point, make preciſely four right Angles, and ſo by *Conſect.* 2. *Definit.* 8. would conſtitute a plane Surface, and not a ſolid Angle. Much leſs therefore could three Heptagons or Octagons, or four Pentagons meet in a ſolid Angle, to form a new regular Body; for thoſe added together would be greater than four right Angles. But now, for the Meaſures of theſe five regular Bodies, take the three following

CONSECTARIES.

I. SINCE a *Tetrahedron* is nothing elſe but a Triangular Pyramid, and an *Octahedron* a double Quadrangular one, their Dimenſion is the ſame as of the Pyramids in *Schol.* of *Definit.* 17. p. 27.

II. The Solidity of an Hexahedron may be had from *Conſect.* 3. *Definit.* 14. p. 21.

III. A Dodecahedron conſiſts of twelve quinquangular Pyramids, and an Icoſahedron of twenty triangular ones, all the Vertexes whereof meet in the Center of a Sphere that is conceived to circumſcribe the reſpective Solids; and conſequently they have their Altitudes and Baſes equal: Wherefore having found the Solidity of one of thoſe Pyramids, and multiplied it by the Number of Baſes (in the one Solid 12, in the other 20) you have the Solidity of the whole reſpective Solids.

DEFINITION XXII.

BESIDES theſe Definitions of the regular Bodies, we may alſo form like Ideas of them from their Geneſis, which particularly *Honoratus Faber* has given us a ſhort and ingenious Syſtem of, in his *Synopſis Geometrica*, p. 149. and afterwards.

p. 34. I. Suppoſe an equilateral Triangle ABD to be inſcribed in a Circle (*Fig.* 44. *N.* 1.) whoſe Center is C, from whence having drawn the Radii C A, C B, C D, imagine them to be liſted up together with the common Center C, and lengthened out, ſo that the Point C aſcending perpendicularly, you have the Lines EA, EB, ED, equal to the Lines AB, BD, DA. After this way there will be a Space generated or made, conſiſting of four equal and equilateral Triangles, which is called a *Tetrahedron.* Hence we ſhall by and by eaſily demonſtrate the Quantity of the Elevation C E, and the Proportion of the Diameter of the Sphere E F, to be circumſcribed to the remaining part C F, and ſo the Reaſon of the *Euclidean* Geneſis, propoſed *Lib.* 13. *Prop.* 13.

II. Much like this, but ſomewhat eaſier to be conceived, is the Geneſis of the *Octahedron*, where by a mental raiſing of C the Center (*Fig.* 44. *N.* 2.) of the Square A B D E inſcribed in a Circle, together with the Semidiameters C A, C B, C D, C E, until being more and more extended, they at length become the Lines A F, B F, D F, E F, all equal among themſelves, and to the Side of the Square A B or B D; and it's manifeſt that by the like Extenſion conceived to be made downwards to G, there will be formed eight equal and regular Triangles, which will all concur in the two oppoſite Points F and G. We might alſo deduce another Geneſis of the *Octahedron* from a certain Section of a Sphere, and alſo give the like of a *Hexahedron* or Cube: but we have already given the eaſieſt, of the one, *viz.* that which is alſo common to Parallelepipedons; and that of the other juſt now given, is ſufficient to our purpoſe.

p. 34.

III. The Generation of the *Dodecahedron* laid down by *Faber*, which is the ſame as *Euclid*'s, we ſhall make eaſy thus: Upon the Side A B of a Cube (*Fig.* 45. *numb.* 1.) let there be ſet, or conceived to be ſet, two Angles *e* A B and *e* B A, each of 36 deg. then the Angle A *e* B being 108 deg. is the Angle of a Pentagon. On the other ſide of the ſaid Line A B make the Angles A B *c*, B A *d*, each of 72 deg. and the Lines B *c*, A *d*, equal to B *e*, or A *e*, and join *c d*: then we ſhall have a regular *Pentagon*, whoſe two Angles A and B lying upon the Side A B of the Cube, which will be extended beyond E F, the middle of the Cube, when lying along the ſame. Now in the other Figure (*n.* 2.) conceive this *Pentagon* to be moved or raiſed up about the Line A B as it were an Axis, until the Side *c d* thereof ſtands pependicularly over the Line E F, the middle of the Cube, and another *Pentagon* equal and ſimilar to it to be apply'd on the oppoſite Side; and ſo ten other ſimilar and

p. 34.

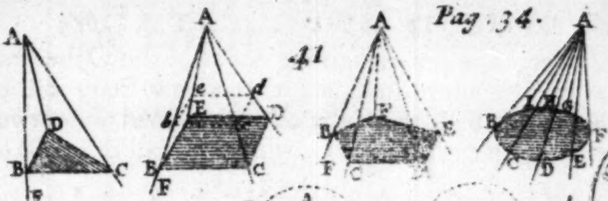
equal ones upon the ten remaining Sides of the Cube; then it will not be difficult to imagine, that a new Body will thus be circumſcrib'd about the Cube comprehended under twelve equal and regular *Pentagons*, which is call'd a *Dodecahedron*.

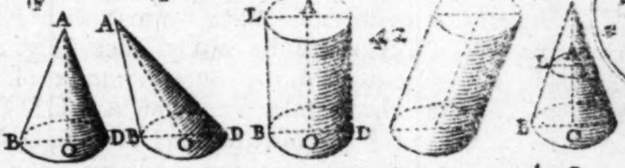
IV. The Generation of an *Icoſahedron* we may with *Faber* eaſily conceive thus: Let there be a Cylinder (*Fig.* 46.) whoſe Altitude FA is equal to the Semidiameter of the Baſe A O, and ſuppoſe the regular Pentagon A B C D E to be inſcribed in it. Draw B F the Side of the Decagon, and let fall the Perpendicular F *a*, and (making the Point *a* the beginning) inſcribe another regular Pentagon equal to the former in the lower Baſe of the Cylinder; then joining A *a*, B *b*, A *e*, B *a*, &c. all round the Cylinder, and it is evident that theſe Lines will form ten Triangles, which are equilateral, and have Sides of every of them equal to the Side A B of the Pentagon, as will be eaſily proved hereafter. Now if we imagine the Lines, O A, O B, O C, O D, O E to be all raiſed up and extended, ſo that the Point O is perpendicularly over the Center, and each of them be equal to the Side A B of the Pentagon, and the ſame is conceived to be done in the lower Baſe of the Cylinder; we ſhall have five more equilateral Triangles above, and five below, equal to the ten former ones; and ſo a Body or Space contain'd under twenty equal and equilateral Triangles, which is call'd an *Icoſahedron*.

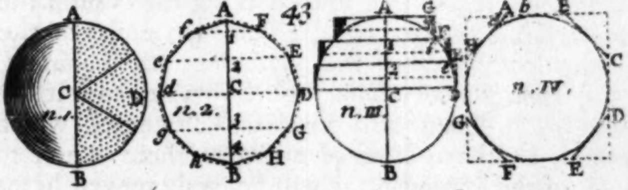
SCHOLIUM.

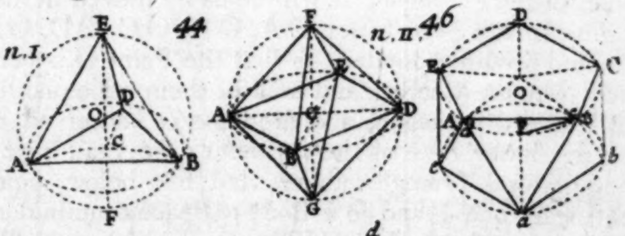
IF any one has a mind to form theſe regular Bodies mechanically of ſtiff Paper, he may do it by duly folding up
p.34, 60. and faſtening together the Schemes (in *Fig.* 47.) conſiſting of equal and regular Planes, &c.

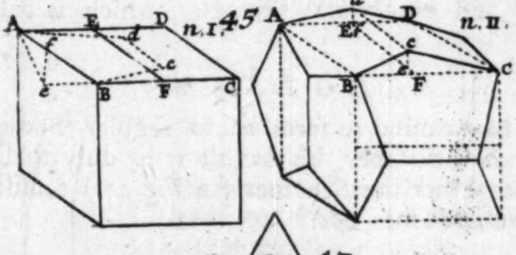
* * *











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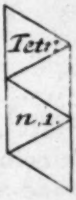
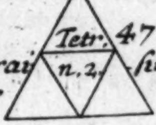
41	27
42	28
43	29
44	32
45	33
46	34
47	ib.





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Schemes for raiſing the five Regular Bodys

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CHAP. II.

Containing the Explication of thoſe Terms, which relate to the Affections of the Objects of the Mathematicks.

DEFINITION XXIII.

EVERY Quantity is ſaid to be either *Finite*, if it has any Bounds or Terms of its Quantity; or *Infinite*, if it has none; or at leaſt *Indefinite*, if thoſe Bounds are not determined, or at leaſt not conſidered as ſo: as *Euclid* often ſuppoſes an *Infinite* Line, or rather perhaps an *Indefinite* one, *i.e.* conſidered without any relation to its Bounds or Ends. By a like diſtinction, (and in reality the ſame with the former) all Quantity is either *Meaſurable*, or ſuch that ſome Meaſure or other, repeated ſome number of times, either exactly meaſures, and ſo equals it, (which *Euclid* and other *Geometricians* emphatically or particularly call Meaſuring;) or elſe is greater; or, on the contrary, *Immenſe*, whoſe Amplitude or Extenſion no *Finite* Meaſure whatſoever, or how many times ſoever repeated, can ever equal. In the firſt Caſe, the Meaſure (*viz.* which exactly meaſures any Quantity) is call'd by *Euclid* an *aliquot* * *Part*, or ſimply a *Part* of the thing meaſured: as *e.g.* the Length of one Foot is an aliquot Part of a Length or Line of ten Foot. In the latter Caſe, the Meaſure (which does not exactly meaſure any Quantity) is called an *Aliquant* Part; as a Line of three or four foot, is an *Aliquant* Part of a Line of ten foot. Now therefore, omitting that perplex'd Queſtion, Whether or not there may be an infinite Magnitude? we ſhall here, reſpecting what is to our purpoſe, deduce the following

* *Lib.*5. *Def.*1.

CONSECTARY.

EVery Meaſure, or Part ſtrictly ſo taken, is to the thing meaſured, or its Whole, as Unity to a whole Number: for that (which is one) repeated a certain number of times, is ſuppoſed exactly to meaſure the other.

DEFINITION XXIV.

IF the ſame Meaſure meaſures two different Quantities (whether the one can exactly meaſure the other or not) thoſe Quantities are ſaid to be abſolutely *Commenſurable*; but if they can have no common Meaſure, they are *Incommenſurable*: Notwithſtanding which, they both retain one to the other a certain relation according to Quantity, which is call'd *Ratio* or *Proportion*, as we ſhall further ſhew hereafter. In the mean while, we have hence, as an infallible Rule to try whether Quantities can admit of a common Meaſure or not, this

CONSECTARY.

THOSE Quantities are Commenſurable, whereof * one is to the other, either as Unity to a whole Number, or as one whole Number to another: for either one of them is the Meaſure of the other, as alſo of itſelf, and then it is to that other, as Unity to a Number by the *Conſect.* of the laſt *Def.* or elſe they admit of ſome third Quantity for a common Meaſure, which will be to either of them ſeparately as Unity to ſome Number: therefore they are one to another, as Number to Number.

p. 35.

* *Euclid. lib.* 10. *Prop.* 5, 6, 7, 8.

DEFINITION XXV.

IF two Quantities of the ſame kind, conſidered as Meaſures one of the other, being applied one to the other, exactly agree, or are exactly equal every way, (as *e.g.* two Squares on the ſame common Side, or two Triangles whoſe Lines, Angles, and Spaces exactly agree and coincide) or at leaſt may be equally meaſured by a common Meaſure applied to both, (as *e.g.* a Square and an Oblong, or a Rhombus; or Triangle, each of whoſe Areas were twenty ſquare Inches, altho' they do not agree in Lines and Angles:) the firſt Quantities may be called *ſimply equal*, and the other *totally equal*, or *equal as to their Wholes.* But if one be greater and the other leſs, then they are *unequal*, and that which exceeds is call'd the *greater*, and that which is deficient, the leſs; and that part by which the leſs is exceeded by the greater, in reſpect to the greater, is call'd *Exceſs*; in reſpect to the leſs, *Defect*; and by a common Name they are call'd the *Difference.* All which, as they are plain and eaſy, ſo they afford us a great many ſelf-evident Truths, which are uſed to be call'd Axioms, as theſe and the like

CONSECTARIES.

I. THE Whole is greater than its Part, whether it be an aliquot or aliquant Part.

II. Thoſe Quantities which are equal to a third, are equal betwixt themſelves.

III. That which is greater or leſs than one of two equal Quantities, is alſo greater or leſs than the other.

IV. Thoſe Things which, being apply'd one to the other, or placed one upon the other, either really or mentally, agree, may be eſteemed as *totally equal:* And, contrariwiſe,

V. Thoſe Things which are *totally equal* will agree, *&c.* To which might be added, ſeveral others which we have already made uſe of, and ſuppoſed as ſuch in the preceding *Definitions.*

DEFINITION XXVI.

A*Ddition*, *Subſtraction*, *Multiplication* and *Diviſion*, are common Affections of all Quantities as well as of Numbers. *Addition* is the Collection of ſeveral Quantities (for the moſt part of one kind) into one Total or Sum; which is either done ſo, that the whole (which is commonly called the *Sum* or *Aggregate*) obtains a new Name, or elſe by a bare Connexion of the Quantities to be added by the Copulative *and*, or the uſual Sign + (*i.e. plus* or more) as for example; 2 Numbers . . . and . . . . (ſuppoſe 3 and 4) added together make the Sum . . . . (*i.e.* 7, or which is the ſame thing $3+4$;) and this Line—— added to this other———— gives the Sum————, which is nothing but the two Lines join'd, or taken together. But now if we would treat of theſe Lines, or any other two Quantities to be added, more generally; by calling the firſt a, and the latter b, we may fitly write their Sum $a+b$.

SCHOLIUM.

HAving thus explained the Term of *Addition*, theſe and the like Axioms emerge of themſelves: *If to equal Quantities you add equal, the Sum will be equal; but if to equal, you add unequal, the Aggregate will be unequal*, &c. Moreover it may not be amiſs to admoniſh the *Tyro* of theſe two things: 1. In Addition may be ſeen the vaſt uſefulneſs of that very ingenious tho' familiar Invention, mentioned in *Definit.* 3. for hereby we may collect into one Sum, p. 2.

not only Tens and Hundreds, but Thouſands, Millions, Myriads, as tho' they were only Units; which we will illuſtrate by an Example.

II. In the *Weigelian* Quaternary Arithmetick, the Addition of Numbers is ſo much the eaſier, by how much the Figures in this Numeration are leſs than in the other; for here you have never but 1 and 2, or 1 and 3, or 2 and 3, or 3 and 3 to add together: whereas in common Addition you muſt ſometimes add 7 and 9, and 8 and 6, &c. together, which Experience proves to be more difficult. But that the reaſon of this Quaternary Addition may be better underſtood, take the following Example, wherein the following Numbers ſet under one another as uſual, are to be added together.

N.B. according to *Weigel.*

```
   1  2  3. 0  2
      2. 3. 2  1.
      1  2  3. 3
   3. 2  1  3. 2.
      2. 0  3  3.
      1  2. 1. 1
      3. 2  1  1
      3. 3. 2  2.
   2  3  1. 0  3
 ________________
    ·     ·
 : 2  3 ·1· 3  2
 ________________
 ·  ·  ·
 3  0  0  1  3  2
```

Beginning at the right Hand, I get all the 4's poſſible in the firſt Column, and writing underneath the 1's remaining, which here are 2, I denote the 4's found; if their Number be leſs than four, under the next Column, with ſo many Dots; but if their Number be four, or twice four, &c. I denote them under the third Column: And proceeding after this manner, and adding together the Dots with the Numbers, write under them after the ſame manner, as partial Sums, and you will have the Sum deſir'd.

N. B. 1. The ſame Sum may be more immediately had, by adding the Number of the 4's found, to the Units of the following Column; or if the Number of the 4's be once, twice, &c. by denoting them by ſo many Dots under the third Column, and afterwards in like manner numbering up the Units of it, when you are come to it, &c.

N. B. 2. This Addition is proved by caſting away of 3's, as the other is by 9's, but yet eaſier.

DEFINITION XXVII.

SUbſtraction is the taking one Quantity from another of the ſame kind; which is ſo perform'd, that either the *Remainder* or *Difference* obtains a new Name, or by a bare

Separation of the Subtrahend by the privative Particle *leſs*, or the uſual Sign — which ſtands for it. As for example . . . or 3, being ſubſtracted from . . . . or 7, the Remainder or Difference is . . . . or 4; and this Line——— ſubſtracted from that —————, leaves —— Now if we would ſignify this more generally, either of theſe Lines, or the Numbers above, or any two Quantities whatſoever that are to be ſubſtracted one from the other, by naming the firſt a, and the latter b, we ſhall have the Remainder $a-b$. Hence are evident theſe and the like Axioms: *If from equal Quantities you ſubſtract equal ones, the Remainders or Differences will be equal.* And here, from this and the *preced. Definit.* it will be worth while to take notice of the following

CONSECTARIES.

I. IF a negative Quantity be added to itſelf conſidered as poſitive (as -3 to $+3$, or $-a$ to $+a$) the Sum will be o; for to add a Privation or Negative is the ſame thing as to ſubſtract a Poſitive; wherefore to join a Negative and Poſitive together, is to make the one to deſtroy the other.

II. If a negative Quantity be ſubſtracted from its poſitive ($-a$ from $+a$) the Remainder will be double of that Poſitive, *viz.* $+2a$. For to ſubſtract or take away a Privation or Negative, is to add that very thing, the Privation of which you take away; for really that which in Words is called the *Addition of a Privation*, is in reality a Subſtraction; and a *Subſtraction* of it, is really an *Addition*: and what is here called a *Remainder*, is indeed a Sum or Aggregate; and what is there called a *Sum*, is truly a *Remainder*. Thus,

III. If the poſitive Quantity ($+a$) be taken from the privative one ($-a$) the Remainder is double the privative one ($-2a$); ſince taking away a poſitive one, there neceſſarily ariſes a new Privation, which will double that you had before. Hence,

IV. You have the Original of the Vulgar Rules in Literal Addition and Subſtraction: *If the Signs of unequal Quantities are different, in the room of Addition you muſt ſubſtract, and in the room of Subſtraction add; and to the Sum or Remainder, prefix the Sign in the firſt place, of the greateſt, in the next, of that from which you ſubſtract: but if the Signs are both the ſame, and the greater Quantity to be ſubſtracted from the leſs, you muſt, on the contrary, ſub-*

ſtract according to the natural Way, the leaſt from the greater, and prefix the contrary Sign to the Remainder : Which Rules you may ſee illuſtrated in the following Examples :

Addition.	*Subtraction.*	
$4b-2a$	from $2a+b$	from $3a+2b$
$3b+5a$	Subſt. $a-b$	Subſt. $2a+3b$
$7b+3a$	R. $a+2b$	R. $a-b$

NOTE.

☞ *Inſtead of the Author's fourth Conſect. as far as it relates to Subſtraction, which may ſeem a little perplex'd, take this general Rule for Subſtraction in Species,* viz. *Change all the Signs of the lower Line, or Subtrahend, and then add the Quantities, and you have the true Remainder.*

SCHOLIUM.

IN this literal Subſtraction, we have not that Conveniency which the Invention of common Figures ſupplies us with, that from the next foregoing Figure we may borrow *Unity*, which in the following Series goes for 10, &c. This is done alſo in Quaternary Subſtraction, only with this difference, that an Unit borrowed here goes only for four. That the Eaſineſs of this Operation may appear, we will add one Example, wherein from this Number, $\dot{1}\dot{2}\dot{3}20023\dot{1}0232$
you are to ſubſtract this, 321012321223

310323323003

Wherever therefore the inferiour Figure is greater than the ſuperiour one, the Facility is much greater here, than in common Subſtraction, becauſe never a greater Number than 3 is to be ſubſtracted out of a greater, than 4 and 2 : but if the inferiour Number be greater than the ſuperiour, you borrow Unity from the left Hand, which is equivalent to 4 ; the reſt is perform'd as in common Subſtraction.

DEFINITION XXVIII.

M*Ultiplication*, generally ſpeaking, is nothing elſe but a complex or manifold Addition of the ſame Quantity, wherein that which is produced is peculiarly call'd a *Product* ; and thoſe Quantities by which it is produced, are called the *Multiplicand* and the *Multiplier* : The firſt denotes

notes the Quantity which is to be multiplied, or added ſo many times to itſelf; and the other, the Number by which it is to be multiplied, or which determines how many times it is to be added to itſelf. The ſame Terms are apply'd moreover to Lines and other Quantities. But here are two things to be chiefly noted: 1. That the Multiplication of one Number by another, or of a Line by a Line, may be conſidered as having a double Event; for the Product may be either of the ſame or a different kind, as, *e.g.* when . . . . 4 is multiplied by 3 . . . the Product may be conſidered either as a Line, thus, ... or as a Plane

. . . .

Surface in this Form, ..., Whence it is alſo named a

. . . .

Plane Number, and the Product is conceived to be formed by the Motion of an erect Line AB, conſiſting of three equal Parts, along another BC, conſiſting of four equal Parts, and conceived as lying along. So alſo the Multiplication of Lines (*e. g.* of the Line A———B by the Line B———C) may be conceived to be ſo performed, that the Product alſo ſhall be a Line, *e.g.* C——————D (concerning the Uſefulneſs of which Multiplication in *Geometry*, we ſhall have occaſion to ſpeak more hereafter;) or ſo, that the Product ſhall be a Plane or Surface, ariſing from the Motion of the erect Line AB, along BC, conceived as lying along; as we have already ſhewn. But as theſe Planes ſo produced, are moſt commonly called *Rectangles*, if the Lines that form them are unequal; and *Squares*, if equal (otherwiſe, the Powers of the given Quantities) in which caſe the Lines that form them are called *Square Roots*: ſo alſo if thoſe Planes are multiplied again into a third Quantity (as into either a Line or Number) there will ariſe *Solids*; and particularly, if that third Quantity be the Root of the Square, the Product is called a *Cube*, &c. The other thing to be noted is, That both theſe ways of multiplying either Numbers, or Lines, are expreſſed by a very compendious, tho' arbitrary way, of Notation, *viz.* by a bare *Juxta-poſition* of the Letters which denote ſuch and ſuch Species of Quantities, as, *e.g.* if for the forementioned Number or Line AB we put a, and for BC, b, the Product will be ab; or if the Multiplicand and Multiplier are equal, as a and a, the Square thence produced, will be aa or a^2; and if this Square be further multiplied by its Root a, then the Cube thence produced will be *aaa* or a^3, &c. Which being premiſed, you have the following

CONSECTARIES.

I. IF a poſitive Quantity be multiplied by a poſitive one, the Product will be alſo poſitive; ſince to multiply, is to repeat the Quantity according as the Multiplier directs: Wherefore to multiply by a poſitive Quantity, is to repeat the Quantities poſitively; as, on the other hand, to multiply by a Privative, is ſo many times to repeat the Privation of that Thing: which we ſhall ſhew further hereafter.

II. Equal Quantities (a and a) multiplied by the ſame b, or contrariwiſe, will give equal Products (ab and ab, or ba.)

III. The ſame Quantity z multiplied by the whole Quantity $(a+b+c)$ or by * all its Parts $a, b, c,$ ſeparately, will give equal Products. Alſo,

IV. The whole $(a+b)$ whether it be multiplied by † itſelf, or by all its Parts (a, b) ſeparately, will give equal Products.

* *Eucl. lib.* 2. *Prop.* 1.

† *Lib.* 2. *Prop.* 2.

SCHOLIUM I.

THE vulgar Praxis of numeral Multiplication, is founded on theſe two laſt Conſectaries: As *e. g.* to multiply 126 by 3; you firſt multiply 6 by 3, then 2, *i.e.* 20 by 3, then 1, *i.e.* 100 by the ſame, and then add each of thoſe partial Products into one Sum: In like manner being to multiply 348 by 23, you firſt multiply each Figure of the Multiplicand by the firſt of the Multiplier (3) and then by the ſecond (2) (*i.e.* 20) &c. which is to be done likewiſe after the ſame manner in quaternary Multiplication; only in this latter, which is more eaſy, you have nothing to reſerve in your Mind, but all is immediately writ down (which might alſo be done in vulgar Multiplication) as may be ſeen by this Example underneath; as alſo the great Eaſineſs of this ſort of Multiplication, beyond the common way, becauſe there is no need of any longer Table, than that we have ſhewn, *pag.* 4.

```
      3 2 1 0 3 2
            1 2 3
    -------------
    2 1 2 3 2 1 2
    1 1 2 1 0 1
  1 2 0 0 0 1 0
  3 2 1 0 3 2
            2
-----------------
1 2 0 0 2 0 3 2 2
```

SCHO-

SCHOLIUM II.

IT is manifeſt from what we have ſaid,

I. If the Baſe of a Parallelogram be call'd b, and its Altitude a, its Area may be expreſſed by the Product ab, by *Conſ.*7; *Definit.* 12. p. 18.

II. If the Baſe of any △ be b or eb, and its Altitude a; its Area will be $\frac{1}{2}ab$ or $\frac{1}{2}eab$, by the 8*th Conſectary* of the ſame *Definition.* ib.

III. If the Baſe of any Priſm or Parallelepipedon, or Pyramid, be $\frac{1}{2}ab$, or ab, and its Altitude c; the ſolid Content of that Priſm will be $\frac{1}{2}abc$, and of the Parallelepipedon abc, by *Conſect.* 3, & 4. *Definit.* 16. and of the Pyramid $\frac{1}{6}abc$, p. 25.
or $\frac{1}{3}abc$, by *Conſ.* 3. *Def.* 17. 27.

DEFINITION XXIX.

Diviſion, in general, is a manifold or complicated Subſtraction of one Quantity (which is called the *Diviſor*) out of another, (which is call'd the *Dividend*) whoſe Multiplicity, or how many times the one is contained in the other, is ſhewn by another Quantity ariſing from that Diviſion; which is therefore call'd the *Quote* or *Quotient*. Here alſo the *Diviſor* is of the ſame or of a different kind with the *Dividend*; *e. g.* of the ſame kind, if the Product above ....(12) be divided by (3) that thence you have the Quotient .... (4) or dividing the aforementioned Line CD by the Line AB, you'll again have the Line BC; but

....
of a different kind, if the plane Number above found ....
....

or the Rectangle ABCD be divided by a Retroduction, or a moving the erect Side AB, by whoſe motion the Rectangle was firſt formed, backward again; that ſo the Line BC may remain alone again. But both theſe kinds of Diviſion, as they have their peculiar Difficulties in Arithmetick and Geometry, which we ſhall further elucidate in their proper places; ſo they may be univerſally and very eaſily perform'd in Species, or by Letters, (which will be ſufficient to our preſent purpoſe) or by a bare Separation of the Diviſor from the Dividend, if it be actually therein included; or by placing the Diviſor underneath the Dividend with a Line between them. Thus if ab be to be divided by b, the Quotient will be a; if by a, the Quotient will be b; but if a or ab be to be divided by c, which Letter, ſince it is not found in the Dividend, cannot be taken out of it, the Quotients are $\frac{a}{c}$ and $\frac{ab}{c}$, *i.e.* a or ab divided by c, after the ſame manner as if 2 were to be divided by 3; which Diviſor, ſince it is not contained in the Dividend, is uſually placed under it, by a ſeparating Line thus $\frac{2}{3}$, or 2 divided by 3.

SCHOLIUM.

HOW difficult common Diviſion is, eſpecially of a large Dividend by a large Diviſor, is ſufficiently known: but how eaſily it is perform'd by Quaternary Arithmetick, we we will barely bring one Example to ſhew. If the Product found in *Schol.* 1. of the preceding *Definition*, *viz.* 1200 203 22 be again to be divided by its Multiplier 123, it may be perform'd after the uſual way, but with much more eaſe, as the following Operation will ſhew; or according to a particular way of *Weigelius*, by writing down the Diviſor, and its double and triple, in a piece of Paper by itſelf, after this way:

1 2 3	3 1 2	1 1 0 1
Diviſor,	*Double,*	*Triple.*

And then moving that piece of Paper to the Dividend, note which of thoſe three Numbers comes neareſt to the firſt Figures of the Dividend; for that barely ſubſtracted, gives the Remainder, and will denote the Quotient to be writ down in its proper place: as the Operation itſelf will ſhew, better than any Words can.

$$\begin{array}{r}
\cancel{1}\;\cancel{1}\cancel{1}\cancel{1}\cancel{1}\\
\cancel{1}\;\cancel{3}\;\cancel{2}\cancel{2}\cancel{2}\cancel{2}\\
\cancel{3}.\cancel{2}.\cancel{3}\cancel{0}\cancel{1}\cancel{1}\cancel{1}\\
\cancel{1}\cancel{2}\;\cancel{0}\;\cancel{0}\cancel{2}\cancel{0}\cancel{3}\cancel{2}\cancel{2}\Big(\;321032\\
\cancel{1}\;\cancel{2}\;\cancel{3}\cancel{3}\cancel{3}333\\
\cancel{1}\;\cancel{2}\cancel{2}\cancel{2}\cancel{2}\cancel{2}2\\
\cancel{1}\cancel{1}\cancel{1}\cancel{1}
\end{array}$$

Thus after *Weigelius*'s way:

$$\begin{array}{r|l|l}
\cancel{1} & & \\
\cancel{2} & \cancel{1} & \\
\cancel{3}\cancel{3} & \cancel{0}.\cancel{1}\cancel{1} & \\
\cancel{1}\cancel{2}\cancel{0}.\cancel{0} & \cancel{2}\cancel{0}\cancel{3}.\cancel{2}\cancel{2} & 321032
\end{array}$$

DEFINITION XXX.

Extraction of Roots is a Species of Diviſion, wherein the Quotient is the Root of the given Square or Cube, *&c.* But the Diviſor is not given, neither is it all along the ſame (as it is in Diviſion) but muſt be perpetually found; and they are ſeveral. And as the Squares of ſimple Numbers 1, 2, 3, &c. *viz.* 1, 4, 9, 16, *&c.* and their Cubes 1, 8, 27, 64, *&c.* may be had immediately out of a Multiplication Table, as alſo their Roots, without any further trouble; and likewiſe in Species, as the Roots of the Square aa or a^2, or of the Cube aaa or a^3, are without doubt a; ſo if the ſquare Root be to be extracted out of de, or the cube Root out of fgm (becauſe the Letters are different, and no one can be taken for the Root) the ſquare Root is commonly noted by this Sign $\sqrt{de}$, the cube Root by this $\surd$C, or $\sqrt[3]{fgm}$, &c. as alſo in Numbers that are not perfect Squares (as *e. g.* 2, 3, 5, 6, 7, 8, 10, 11, 15, 17, 19, *&c.*) we can no otherwiſe expreſs the ſquare Roots, than after this manner $\surd 2$, $\surd 7$, $\surd 19$, *&c.* and in thoſe that are not perfectly cubical (as all between 1, 8, 27, 64, *&c.*) we can only expreſs their cube Roots after ſome ſuch manner, $\surd c.\,7$, or $\sqrt[3]{7}$. $\surd c.\,61$, or $\surd 61$, *&c.* Which Forms of Roots in ſpecious Computation, we call *Surd Quantities*, in vulgar Arithmetick *Surd Numbers*, *i. e.* ſuch as cannot be perfectly expreſſed by any Numbers; altho' we have Rules at hand to determine their Values nearer and nearer *ad Infinitum.*

Theſe Rules accommodated to ſquare and cube Numbers, *&c.* which otherwiſe are more difficult to be comprehended, appear plain and eaſy to him, who multiplies a Root expreſſed by two Letters a and b (called therefore commonly a Binomial) firſt quadratically, then cubically, *&c.* For he will have as

CONSECTARIES,

I. THE Square of any aſſumed Root, as alſo, *Prop.* 4. *lib.* 2. *Eucl.* and at the ſame time a general Rule for extracting the ſquare Root, all expreſſed in theſe few Notes:

$$aa + 2ab + bb.$$

II. The Cube of the ſame Root, and at the ſame time a Rule for extracting any cube Root, contained in this Expreſſion,

$$a^3 + 3aab + 3abb + b^3.$$

SCHOLIUM.

WHICH that we may the more plainly ſhew, eſpecially as far as it relates to the Rules of Extraction, conſider, 1. That the Root of the Square $aa+2ab+bb$ is already known (for we aſſumed for the Root the Quantity $a+b$) ſo that now we are to ſee which way this Root is to be obtain'd out of that Square by Diviſion. It will preſently appear, that the firſt part of the Root a, will come out of the firſt Part of the Square aa, and the other Part b muſt be obtained out of the Remainder $2ab+bb$; and ſo as there are two Parts in the Root, the Square muſt be diſtinguiſhed, as it were, into two Claſſes, each of which gives a particular Part of the Root. Then it is manifeſt, that the firſt Part of the Root, *viz.* a, may be obtain'd out of the Square aa by a ſimple Extraction. Now it is evident, If I would have by Diviſion the other Part of the Root, the next following Part of the remaining Claſſis muſt be divided by $2a$, the double of the Quotient juſt now found; and that nothing ſhould remain after this Diviſion (for now we have the whole Root $a+b$) you muſt not only ſubſtract the Product of the Diviſor and this new Quotient, but alſo the Square of this new Quotient: Which is the vulgar Method and Rule for the Extraction of ſquare Roots taught in common Arithmetick.

Likewiſe if you would extract the Root of the abovemention'd Cube, which we already know, having formed it from $a+b$, it is manifeſt, that the firſt Part of the Root a will come out of the firſt Part of the Cube a , and the other b, muſt be obtained out of the Remainder $3aab+3abb+b^3$, and ſo, as there are two Parts in the Root, the Cube muſt be diſtinguiſhed, as it were, into two Claſſes, each of which will give a particular Part of the Root. Now it is manifeſt, the firſt Part of the Root a is obtained by the ſimple Extraction of the Root out of the Cube aaa. It is moreover evident, if I would have by Diviſion the other Part of the Root b, the next remaining Part muſt be divided by $3aa$ (the triple Square of the precedent Quotient, or thrice the precedent Quotient multiplied by itſelf) and, that nothing ſhould remain after this Diviſion (for now we have the whole Cube Root $a+b$) you muſt not only ſubſtract the Product of the Diviſor from the remaining Dividend, and the new Quotient ($3aab$) but alſo the Product of the Square of the new Quotient, and thrice the precedent Quotient ($3abb$) and moreover

over the Cube of that new Quotient b^3 : Which is the Method of extracting Cube Roots in Vulgar Arithmetick.

SCHOLIUM II.

FROM what we have ſaid, you have alſo the Reaſon of another Rule in Arithmetick, which teaches how to approach continually nearer and nearer to the Square and Cube Roots of Numbers that are not exact Squares and Cubes; *viz.* by adding to the given Number perpetually new Claſſes of Cyphers or 0's, two at a time to the Square, and three to the Cube, and ſo continue on the Operation as before; which will add Decimal Parts to the Integrals before found; and the next Operation (if you add a ſecond Claſs of Cyphers) will exhibit Centeſimal Parts, and ſo on *ad infinitum.* For Example, if I would have the Square Root of the Number 2 pretty near, I can aſſign no nearer whole Number than 1: But by adding a new Claſs of two Cyphers, *i.e.* multiplying the given Number by 100 (whereby the Root is multiplied by 10) you'll have 14, nearly the Root of 200, that it, $\frac{14}{100}$ or 1 $\frac{4}{10}$ much nearer the Root of two than the former; and thus you may always come nearer and nearer *ad infinitum*, but never to an exact Root. For if you could have the exact Root of 2, or 3, or 5, *&c.* in any Fraction whatſoever, that Fraction muſt be of ſuch ſort, that its Numerator and Denominator being ſquared, the Fraction thence ariſing muſt exactly equal 2, or 3, or 5, *&c.* that is, its Numerator muſt be exactly double, or triple, or quadruple, *&c.* of the Denominator; which can never be, becauſe both are Squares, and in a Series of Squares no ſuch thing can happen. Hence you have theſe

CONSECTARIES.

III. THAT it is a certain Mark of Incommenſurability, if one Quantity is 1, and the other $\sqrt{2}$, or $\sqrt{3}$, or $\sqrt{5}$, *&c.*

IV. That theſe Sorts of Quantities are notwithſtanding commenſurable in their Powers, *i. e.* their Squares are as 1 and 2, or as a Number to a Number.

V. Thoſe Quantities which are to one another, as 1 and $\sqrt{\sqrt{2}}$, or as $\sqrt{2}$ and the $\sqrt{\sqrt{3}}$ are incommenſurable in Power alſo. Which being rightly underſtood, you may eaſily comprehend ſeveral (*a*) Propotions of *lib.* 10. *Eucl.* eſpecially after ſome few things premiſed concerning Ratio and Proportion.

(*a*) *Eucl. l.* 10. *Prop.* 9, 10, 11, 12, *&c.*

SCHOLIUM III.

FROM what we have ſhewn, it may eaſily be concluded, that to any propoſed Quantity whatſoever, which *Euclid* calls (*b*) *Rational*, and for which we may always put 1, there may be ſeveral others both commenſurable and incommenſurable, and that either ſimply or in power ſo: thoſe which are commenſurable to a rational given Quantity, either ſimply, or only in Power (which, *e.g.* are to it, as 2, 3, 4, &c. $\frac{1}{2}$, ^{1}, $\frac{1}{4}$, &c. or as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{^1}$, $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{4}}$, &c. are called alſo *Rational*: but thoſe which are incommenſurable both ways (*i.e.*) both ſimply and in power) as $\sqrt{\sqrt{2}}$, $\sqrt{\sqrt{\frac{1}{3}}}$, &c.) are called *Irrational*. In like manner the Square of a given Rational Quantity (as 1) is called Rational, and Quantities commenſurable to it (as the Squares of 2, 3, 4, 5, &c. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. are called alſo *Rational*: but incommenſurable ones ($\sqrt{\sqrt{2}}$, $\sqrt{\sqrt{5}}$, &c.) *Irrational*, and the Sides and Roots of theſe more *Irrational*.

(*b*) *Lib.* 10. *Def.* 5, 6, 7, 8, 9, 10, 11.

DEFINITION XXXI.

ANY Quantities of the ſame kind whatſoever, whether commenſurable or incommenſurable, equal or unequal, admit of a twofold Reſpect or Relation of their Magnitude; one whereof, when only the Difference or Exceſs of one above another is reſpected (as 10, which is 3 more than 7) is called an *Arithmetical Ratio* or *Reſpect*; the other, wherein reſpect is rather had to the Amplitude, whereby one is contained once, or a certain Number of Times in the other (as three is contained thrice in 10 and $\frac{1}{3}$ Part more) is called *Geometrical Ratio*, or, by way of Emphaſis, only *Ratio*, and by others *Proportion*: and this Ratio or Proportion, if the leſs be exactly contained a certain Number of Times in the greater (as 3 in 6, or 4 in 12) is generally called, on the part of the greateſt Term, *Multiple*, and on part of the leſs, *Submultiple*; and particularly in the firſt Example *double*, when 6 is taken in reſpect to 3, and *ſubduple* when 3 is taken in reſpect to 6; in the other, *triple* and *ſubtriple*, &c. If the leſs be contained in the greater once or more times, Unity only remaining over and above (as 3 in 4, and 4 in 9) the *Ratio* or *Proportion* is called *Superparticular* and *Subſuperparticular*, and is noted by the Terms *Seſqui* and *Subſeſqui*, joining the ordinal Names of the leſſer Term; as the *Ratio* of 4 to 3 is called *Seſquitertian*, and contrariwiſe

Subſeſquitertian; the *Ratio* of 9 to 4 is called double *Seſquiquartan*, and contrariwiſe *Subduple Subſeſquiquartan*, &c. If, laſtly, the leſs be contained in the greater once, or a certain Number of times, ſeveral Units remaining over and above, it is commonly called *Superpartient Ratio*, and is expreſſed by the word *Super* or *Subſuper*, joined with the adverbial Name of the remaining Parts, and the ordinal Name of the leſſer Term: thus, *e.g.* the *Ratio* of 7 to 4 is called *Supertriquartan*, 12 to 5 *double Superbiquintan*, &c. Moreover, the Quotient which ariſes by the Diviſion of the greater Term by the leſs, and is commonly expreſſed in the ſame words, is alſo commonly called the *Name of a Ratio*, (*e.g.* 2 is the Name of the *Ratio* of 6 to 3, $2\frac{1}{4}$ of 9 to 4, or contrariwiſe, *&c.*) as alſo the Quotient ariſing by the Diviſion of the Conſequent by the Antecedent (as $\frac{1}{2}$ in the firſt Caſe, $\frac{4}{9}$ in the latter) by which Name the antecedent Term of the *Ratio* being multiplied, produces its Conſequent; which is evident by naming any *Ratio*, e or i, or o, &c. Thus if the antecedent Term be called a or b, &c. the conſequent Terms may be rightly called ea or eb, oa or ob, &c. and becauſe in an Arithmetical Relation we only reſpect the Exceſs of the firſt above that following it, or of the following Term above the foregoing one (which may be called x or z, &c.) if the Antecedent (which may be called a or b) be the leſs, the Conſequent may properly be called $a+x$ or $b+z$; but if it be the greater, the other will be $a-x$ or $b-z$.

CONSECTARIES.

I. We may hence readily infer, that if the Diameter of any Circle be called a, the Circumference may be called ea (for whatever the Proportion is between them, it may be expreſſed by the Letter e) and the Area, according to *Conſect.* 2. *Defin.* 15. will be $\frac{1}{4}eaa$. p. 22.

II. If you put $\frac{1}{4}eaa$ for the Baſe of any Cylinder or Cone, and b for the Altitude, the Solidity of that Cylinder may be rightly expreſſed by $\frac{1}{4}eaab$, by *Conſect.* 5. *Defin.* 16. p. 25.
and of the Cone by $\frac{1}{12}eaab$, by *Conſect.* 4. *Definit.* 17. 27.

DEFINITION XXXII.

As the Identity (or Sameneſs) of ſeveral Geometrical *Ratio's* is uſually called *Geometrical Proportionality*, or emphatically *Proportion*; ſo the Similitude (or Likeneſs) of ſeveral *Arithmetical Ratio's*, is deſervedly called *Arithmetical Proportionality*, or by a particular Name *Progreſ-*

ſion; and conſequently thoſe *Progreſſionals*, or *Arithmetical Proportionals*, which exceed one another by the ſame Difference, either uninterruptedly or continually, as 2, 5, 8, 11, 14, &c. aſcending, or 30, 28, 26, 24, 22, 20, &c. deſcending; or interruptedly, as 2 and 5, 7 and 10, 11 and 14, &c. aſcending; or 30 and 26, 24 and 20, 18 and 14, &c. deſcending: For which, and all others in what Caſe ſoever, we may univerſally put this (or ſuch like) continual Progreſſion, *v.g.* a, $a+x$, $a+2x$, $a+3x$, &c. aſcending; or, a, $a-x$, $a-2x$, $a-3x$, &c. deſcending, but in an interrupted Progreſſion, *v.g.* b and $b+z$, c and $c+z$, d and $d+z$, &c. aſcending, or b and $b-z$, c and $c-z$, d and $d-z$, &c. deſcending. Whence you have this

CONSECTARY.

ANY Difference being given, the following Terms of the Progreſſion, continually proceeding from any aſſumed or given one, may be found; as alſo ſeveral Antecedents that interruptedly follow the given or aſſumed ones, *viz.* by adding or ſubſtracting the given Difference to or from the former Terms.

DEFINITION XXXIII.

IN like manner, ſince *Ratio's* are ſaid to be the ſame, which have the ſame Name, thoſe Quantities will be *proportional* which continually aſcend by *Ratio's* of the ſame Denomination, as 2, 4, 8, 16, 32, 64, &c. or deſcend, as 81, 27, 9, 3, 1. the former by the Denomination 2, and the latter by 3; or that aſcend interruptedly, as 2, 4; 3, 6; 5, 10, &c. or deſcend, as 40, 10; 28, 7; 20, 5; 8, 2, &c. Whence you have theſe

CONSECTARIES.

I. TWO Terms being given, or only one with the Denomination of the *Ratio* (*e.g.* the Term 2 with 3, the Denomination of the *Ratio*, or univerſally the firſt Term a with e, the Denomination of the *Ratio*) it will be eaſy to find as many more Terms of the Geometrical Progreſſion or Proportion as you pleaſe, *viz.* by always multiplying the Antecedent by the Denomination of the *Ratio*, that you may have 2, 6, 18, 54, &c. or a, ea, e^2a, e^3a, &c. in continued, or 2 and 6, 4 and 12, 5 and 15, &c. and aea, beb, ded, &c. in diſcontinued or interrupted Proportion.

Thus having rightly underſtood what we have ſaid in this
p. 48. 33d, and the 31ſt *Definition*, there will follow theſe *Corollaries*
as ſo many Axioms.

II. That equal Quantities have the ſame Proportion to the ſame Quantity (*a*), and the ſame has the like to equal Quantities.

(*a*) Eucl. l. 5. Prop. 7.

III. But a greater Quantity has a greater *Ratio* to the ſame (*b*) than a leſs, and the ſame has a greater Proportion to a leſs Quantity than to a greater.

(*b*) Prop. 8.

IV. On the contrary, thoſe that have the (*c*) ſame Proportion to the ſame Quantity, and that likewiſe the ſame to them, are equal.

(*c*) Prop. 9.

V. But that which bears a (*d*) greater Proportion to the ſame, is greater; and that to which the ſame bears a greater Proportion, is leſs.

(*d*) Prop. 10.

VI. Proportions equal to any Third (*e*). are alſo equal among themſelves, &c.

(*e*) Prop. 16.

DEFINITION XXXIV.

HERE remain two things to be taken notice of; the firſt whereof is this; If any whole (Quantity) be ſo divided into two equal (*f*) Parts, that the Whole, the greater Part, and the leſs, are in continual Proportion; that (Whole) is ſaid to be cut in *extreme and mean Ratio*. 2. In a continual Series of Proportionals (*e.g.* 2. 4. 8. 16. 32, &c. or a, ea, e^2a, e^3a, e^4a, &c.) the *Ratio* of the firſt Term to the (*g*) third (2 to 8, or a to e^2a) is particularly called *Duplicate*, and of the firſt to the fourth (16 or e^3a) *Triplicate*, &c. of that *Ratio* which the ſame firſt Term has to its ſecond, or any other Antecedent of that Series to its Conſequent: But generally theſe Duplicate and Triplicate *Ratio's*, &c. as others alſo of the firſt Term to the third or fourth, or Proportions continually cohering together (whether they are the ſame as in the foregoing Examples, or different, as in theſe, 2, 4, 6, 18, or a, ea, eia, $eioa$, &c. *viz.* if the Name of the firſt *Ratio* be e, of the ſecond i, of the third o, &c.) I ſay the *Ratio's* of the firſt Term (2 or a) to the third (6 or eia) or to the fourth (18 or $eioa$) are ſaid to be *compounded* of the continual intermediate *Ratio's*.

(*f*) Euclid. Definit. 3. lib. 6.

(*g*) Eucl. Definit. 10. l. 5.

Now, from our general Example, what *Euclid* ſays, is manifeſt.

CONSECTARY I.

THAT the Denomination of a compounded *Ratio* ariſes from the Multiplication of the Denominations of the given ſimple (*a*) *Ratio's*; as the Denomination of the *Ratio* compounded of both, (*viz.* *a* to *eia*) is produced by multiplying the Denomination of the firſt *Ratio e*, by the Denomination of the ſecond *Ratio i*, and the Denomination of the *Ratio* compounded of the three (*viz.* *a* to *eioa*) is produced by the Denomination of the firſt *Ratio e*, multiplied by the Denomination of the ſecond *Ratio i*; and the Product of theſe by the Denomination of the third *Ratio*, *o*, &c.

(*a*) Eucl. l.6. Def. 5.

CONSECTARY II.

SO that it is very eaſy, after this way, having never ſo many *Ratio's* given, whether continued (as 2 to 3, 3 to 6, or *a*, *ea*, *eia*) or interrupted or diſcreet (as 2 to 3, and 5 to 10, or *a* to *ea*, and *b* to *ib*) to expreſs their compounded *Ratio*: In the firſt Caſe it is eaſily obtained by the bare Omiſſion of the intermediate Term or Terms (2 to 6, or *a* to *eia*:) and in the other, by multiplying firſt of all the Names of the compounding *Ratio's* among themſelves (1 $\frac{1}{2}$ and 2, *e* and *i*) and by the Product (3 or *e i*) as the Name of the *Ratio* compounding the firſt Term (2 or *a*) that you may have the other (6 or *e i a*) or (if any one had rather do ſo in this latter Caſe) by turning the diſcreet or interrupted *Ratio's* into continued ones (by making as 5 to 10 in the ſecond *Ratio*, ſo is the Conſequent of the firſt 3 to 6, or as *b* to *i b*, ſo *e a* to *e i a*) and then by referring the firſt 2 to the third 6, or the firſt *a* to the third *e i a*, &c. In a word therefore, any Duplicate *Ratio* may be appoſitely expreſſed by *a* to *e a*, and Triplicate to *a* by $e^2 a$, the one immediately diſcernible by a double, the other by a triple Multiplication into itſelf; as you may alſo commodiouſly denote others compounded, *e. g.* of two, by *a* to *e i a*, of three, by *a* to *e i o a*, &c.

SCHOLIUM.

WE will here advertiſe the Reader, that tho' the Names of duplicate and triplicate *Ratio's*, &c. are chiefly appropriated to Geometrical Proportionality, yet the Moderns have alſo accommodated them to Arithmetical alſo; as *e.g.* That *Arithmetical Progreſſion* is called *Duplicate*, whoſe Terms are the Squares of Numbers arithmetically proportional (*e.g.* 1, 4, 9, 16, 25, *&c.*) and *Triplicate*, whoſe Terms are Cubes, *&c.* as 1, 8, 27, 64, *&c.*

DEFINITION XXXV.

AND now at length we may underſtand what Magnitudes Geometers particularly call *like*, or *ſimilar*: Whereas in general, one Number may be ſaid to be like another, one right Line like to another, one obtuſe Angle to another, a Triangle to a Triangle, and the like; but an Acute Angle is not like an Obtuſe one, nor a Triangle like a Parallelogram, or a right Line like a Curve one, or a Square like an Oblong, *&c.* Yet among thoſe Figures, which may after that rate in general be ſaid to be like, there is notwithſtanding a great deal of Diſſimilitude; therefore, in a ſtrict Senſe, we call only thoſe right-lined Figures *ſimilar* or *like* *, which have each Angle of the one equal to each Angle of the other (as A and 𝔄, B and 𝔅, C and ℭ, &c. *Fig.* 48.) and the Sides about thoſe equal Angles proportional, *viz.* as B A to A C, ſo 𝔅𝔄 to 𝔄ℭ &c. † and among Solid Figures thoſe are ſaid to be ſimilar, each of whoſe Planes are reſpectively ſimilar one to the other, and equal in Number in each Figure; as *e.g.* the Plane A C ſimilar to the Plane 𝔄ℭ, and C G to ℭ𝔊, &c. and ſix in Number in each Figure.

p. 60.

* *Euclid. Def.* 1. *l.* 6.

† *Def.* 9. *l.* 9.

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BOOK I.

SECT. II.

Containing ſeveral Propoſitions demonſtrated from the foregoing Foundations.

CHAP. I.

Of the Compoſition and Diviſion of Quantities.

PROPOSITION I.

THE Sum and Difference of two unequal Quantities added together, make double of the greateſt.

Demonſtration.

Suppoſe a to be the greateſt, and b the leaſt, then will their Sum be $a+b$

And their Difference $a-b$

Their Sum $2a$, by *Conſectary* 1.

p.39. *Definition* 27. *Q. E. D.*

CONSECTARY.

* Eucl. lib. 1. Prop. 13.

p.9. HENCE, by a bare Subſumption *, you have the Truth of *Conſect.* 1. *Definit.* 8. *viz.* that two unequal contiguous Angles, ACD, ACE,

p.60. on the ſame right Line (*Fig.* 49.) that is, (by calling the right Angle BCD or BCE, and the Difference between the one and the other $a+b$ and $a-b$) make $2a$, *i. e.* are equal to two right Angles.

PROPOSITION II.

If the Difference of two unequal Quantities be ſubſtracted from their Sum, the Remainder will be double of the leaſt.

Demonſtration.

If from the Aggregate or Sum $a+b$
You ſubſtract the Difference $a-b$

The Remainder will be $0+2b$. by *Conſectary* 2. *Definit.* 27. *Q. E. D.* p. 39.

PROPOSITION III.

But if the Sum or Aggregate be ſubſtracted from the Difference, the Remainder is ſo much leſs than nothing, as is the double of the laſt Quantity.

Demonſtration.

For if from the Difference $a-b$
You ſubſtract the Sum or Aggregate $a+b$

The Remainder will be $0-2b$, by *Conſectary* 3. of the aforeſaid *Definition.* *Q. E. D.* p. 39.

PROPOSITION IV.

If a Poſitive Quantity be multiplied by a Negative one, or contrariwiſe, the Product will be a Negative Quantity.

Expoſition.

If $a-b$ *is to be multiplied by* c, *it is certain, that* a *multiplied by* c, *makes* ac *a Poſitive Quantity, by* Conſect. 1. Definit. 28. *Moreover,* b *multiplied by the ſame* c (*a Negative by a Poſitive*) *will make* $-bc$; *and ſo the whole Product of* $a-b$ *by* $+c$, *will be* $ac-bc$. p. 42.

Demonſtration.

Suppoſe $a-b=e$; therefore ec will be $=$ to the Product of $a-b$ by c: and ſince $a-b$ is $=e$ by the Hypoth. adding on both Sides b, you'll have $a=e+b$. by *Schol. Definit.* 26. and multiplying both Sides by c, there will ariſe $ac=ec+bc$, p. 37.

p. 42. by *Conſect.* 2. *Defin.* 28. and by further ſubſtracting bc from each Side, you'll have $ac-bc=ec$, that is, equal to the Product of $a-b$ by c. *Q. E. D.*

CONSECTARY.

SINCE $ac-bc$ is the Product of $a-b$ by c, it is manifeſt alſo, that if $ac-bc$ be divided by c, you'll have $a-b$ for the Quotient; and ſo always a Poſitive Quantity (as ac) divided by a Poſitive one c, will give a Poſitive Quotient; but a Negative Quantity $-bc$ divided by a Poſitive one c, will give a Negative Quotient.

PROPOSITION V.

IF a Negative Quantity be multiplied by a Negative one, the Product will be Poſitive.

Expoſition.

Suppoſe $a-b$ is to be multiplied by $-c$; it is certain, that a multiplied by $-c$ will give the Negative Quantity
p. 55. $-ac$, by *Prop.* 4. but $-b$ multiplied by the ſame $-c$ will produce $+bc$, and ſo the whole Product will be $-ac+bc$.

A Demonſtration like the former.

Suppoſe $a-b=e$, then will $-ec=$ the Product of $a-b$ by $-c$: and ſince $a-b$ is $=e$, adding b on both Sides, you
p. 37. will have $a=e+b$, by *Schol. Definit.* 26. and multiplying both Sides by $-c$, you'll have $-ac=-ec-bc$, by
p. 55. 42. *Prop.* 4. and *Conſect.* 2. *Definit.* 28. and by adding bc on both Sides, you'll have $-ac+bc=-ec$, i. e. equal to the Product of $a-b$ by $-c$. *Q. E. D.*

CONSECTARIES.

I. SINCE therefore $cb-ac$ ariſes from $a-b$ into $-c$, it is manifeſt, that if $-ac+bc$ be divided by $-c$, you will again have $a-b$, and conſequently a Negative Quantity, divided by a Negative one, will give a Poſitive Quotient; but a Poſitive Quantity $+bc$, divided by a Negative one $-c$, will give a Negative Quotient $-b$.

II. We have therefore the Foundation and Demonſtration of the Rules of ſpecious Computation, in the Multiplication and Diviſion of compounded Quantities, *viz.* the ſame Signs multiplied together (as + by + or — by —) give +; but different (as + by — or — by +) give the Sign —:

Which Rules the following Examples will illuſtrate, as alſo ſeveral others we ſhall meet with in the following Chapter.

MULTIPLICATION.

$$\begin{array}{l} a+b \\ a-b \\ \hline aa+ab \\ \quad -ab-bb \\ \hline aa-bb \end{array} \qquad\qquad \begin{array}{l} a-b \\ c-d \\ \hline ac-cb \\ \quad -ad+bd \\ \hline ac-cb-ad+bd \end{array}$$

$$\begin{array}{l} 3d-4e+2f \\ 5d-g \\ \hline 15dd-20de+10df \\ \qquad -3dg+4ge-2gf \\ \hline 15dd-20de+10df-3dg+4gc-2gf. \end{array}$$

DIVISION.

$$\begin{array}{ll} \text{Divide } aa-bb & \\ \text{by } a+b & (a-b \end{array} \;\Bigg|\; \begin{array}{ll} \text{Divide } ac+ad-bc-bd & \\ \text{by } c+d & (a-b \end{array}$$

CHAP. II.

Of the Powers of QUANTITIES.

[Containing moſt part of the ſecond Book of *Euclid*, after a compendious way; and the Appendix of *Clavius* to *lib.* 9. *Prop.* 14.]

PROPOSITION VI.

IF *any whole Quantity be divided into two Parts* *, *the Rectangle contained under the Whole, and one of its Parts, is equal to the Square of the ſame Part, and the Rectangle contained under both the Parts.*

* Eucl. lib. 2. Prop. 3. A. C. alſo the third.

Demonſtration.

$$\begin{array}{lcl} \text{Let } a+b \text{ repreſent the Whole} & & a+b \text{ the Whole.} \\ \quad b \text{ one Part of it,} & \text{or} & a \text{ the other Part.} \\ \hline ab+bb \text{ the Rectangle,} & & aa+ab \text{ the Rectangle.} \\ \quad \text{(See } Fig. \text{ 50.)} & & Q.\,E.\,D. \end{array}$$

p. 60

PRO-

PROPOSITION VII.

* *Eucl. Prop. 4. lib. 2.*

*IF a whole Quantity be divided into any two Parts *, the Square of the Whole is equal to the Squares of both thoſe Parts, and two Rectangles contained undr them.*

Demonſtration.

This is evident from the laſt Propoſition, and may moreover thus appear further.

$$\begin{array}{lr} \text{Let the Parts be } a \text{ and } b\text{, then the whole will be} & a+b \\ \text{Which if you multiply by itſelf} & a+b \\ \hline & aa+ab \\ & ab+bb \\ \hline \text{You have the Square} & aa+2ab+bb \\ \text{(See } Fig.\ 51.\ N.\ 1.\text{)} & Q.E.D. \end{array}$$

p. 60.

CONSECTARIES.

p. 45. 60.

I. HENCE you have the Original Rule for extracting of Square Roots, as we have ſhewn after *Definition* 30. and here have further illuſtrated in the *Scheme* N°. 2.

II. Hence it naturally follows, that the Square of double any Side, is quadruple of the Square of that Side taken ſingly.

III. Hence alſo you have the Addition of ſurd Numbers, or in general of ſurd Quantities, by help of the following Rule (ſuppoſing in the mean while their Multiplication: *Suppoſe theſe two Surds* $\sqrt{8}$ and $\sqrt{18}$, or more generally $\sqrt{75aa}$ and $\sqrt{27aa}$ *are to be added together; firſt add their Squares* 8 *and* 18, *&c. then double the Rectangle* ($\sqrt{144}$) *that is, multiply it by the* $\sqrt{4}$, *and then having extracted the Square Root of this double Rectangle* $\sqrt{576}$, which is 24, *and added it to the Sum of the firſt Squares* (26) *the Root of the whole Sum* (50) *viz.* $\sqrt{50}$, *is the Sum of the two Surd Quantities firſt propoſed.*

SCHOLIUM.

BUT if it happens that the Root of the double Product, or Rectangle, cannot be expreſſed by a Rational Number (as, when the propoſed Quantities are Surds, as $\sqrt{3}$ and $\sqrt{7}$, to whoſe Squares $3+7$, *i. e.* 10, you muſt add the double Product of $\sqrt{7}$ by $\sqrt{3}$, *i.e.* $\sqrt{84}$. (which cannot be expreſſed by a Rational Number) then that double Product

muſt be joined under a Surd Form, or Radical Sign, to the Sum of the Squares (thus, *viz.* $10 + \sqrt{84}$) and to this whole Aggregate prefix another Radical Sign, thus, $\sqrt{10 + \sqrt{84}}$; or elſe you may only ſimply join the Surd Quanties propoſed by the Sign + thus, $\sqrt{3} + \sqrt{7}$. Here alſo you may note, that the two *Surd Quantities* propoſed in the firſt Caſe of *Conſect.* 2, are called *Communicants*; in the other Caſe p.58.
of this *Scholium*, *Non-Communicants*: For in the former Caſe each Quantity under the Radical Sign may be divided by ſome Square, and have the ſame Quotient, (*e.g.* 8 and 18, may be divided, the firſt by 4, the other by 9, and the Quotient of both will be 2; likewiſe $75aa$ and $27aa$ may be divided, the one by $25aa$, and the other by $9aa$, the Quotient of both being 3; and then if the Quotient of both be left under the Radical Sign, and the Root of the dividing Square ſet before it, the ſame Quantities will be rightly expreſſed under this Form: $2\sqrt{2}$ and $3\sqrt{2}$, alſo $5a\sqrt{3}$ and $3a\sqrt{3}$; and then the Addition is eaſy, *viz.* only collecting or adding together the Quantities prefixed to the Radical Sign; ſo that the Sums of the one will be $5\sqrt{2}$, and of the other $8a\sqrt{3}$, which are indeed the ſame we have ſhewn in *Conſect.* 3. For if contrariwiſe we ſquare the Quantities that ſtand without, or are prefixed to the Radical Sign, and then ſet thoſe Squares (25 and $64aa$) under the Radical Sign, multiplied by the Number prefixed to it, you'll have for the one $\sqrt{50}$, and for the other $\sqrt{192aa}$ (*Conſect.* after *Schol.* p.82
Prop. 22.

PROPOSITION VIII.

IF any whole Quantity (viz. *Line or Number*) *be divided* (a) *into two equal Parts, and two unequal ones; the Rectangle under the unequal ones, together with the Square of* (*the intermediate Part, or*) *the Difference of the equal Part from the unequal one, is equal to the Square of half the whole Quantity.*

(a) *Euclid.* & *Clav.* 5.

An Univerſal **Demonſtration.**

Suppoſe the Parts to be a and a, and the Whole $2a$; let one of the unequal Parts be b, the other will be $2a - b$, and the Difference between the equal and unequal Part will be $a - b$.

$$\begin{array}{lr} \text{The unequal ones} & 2a - b \\ & +b \\ \hline \text{Rectangle} & 2ab - bb \end{array} \qquad \begin{array}{lr} \text{Difference} & a - b \\ & a + b \\ \hline & aa - ab \\ & -ab + bb \\ \hline & aa - 2ab + bb\,\square \end{array}$$

The Sum of theſe two Products will be aa (the other Parts deſtroying one another.) *Q. E. D.* (*Vid. Fig.* 52.)

PROPOSITION IX.

IF *to any whole Quantity, divided into two equal Parts, you* * *add another Quantity of the ſame kind, the Rectangle or Product made of the Whole and the Part added, multiplied by that Part added, together with the Square of the half, will be equal to the Square of the Quantity compounded of that half, and the Part added.*

* Eucl. & Clav. 6.

Demonſtration.

Let the Whole be called $2a$, the Part added b; then the Quantity compounded of the Whole and the Part added, will be $2a+b$; and that compounded of the half and the Part added $a+b$.

The Quantity compounded of the Whole, and the Part

$$\begin{array}{lr|r} \text{added, is} \quad 2a+b & \text{the half } a & \text{Comp. } a+b \\ \text{The Quantity added} \quad b & a & a+b \\ \hline \text{The Product} \quad 2ab+bb\ \square & \text{half } aa & ab+bb \\ & & aa+ab \\ \hline \end{array}$$

rentrae 4

The Sum $2ab + bb + aa = \square\, aa + 2ab + bb$

(*Vide Fig.* 53.) *Q.E.D.*

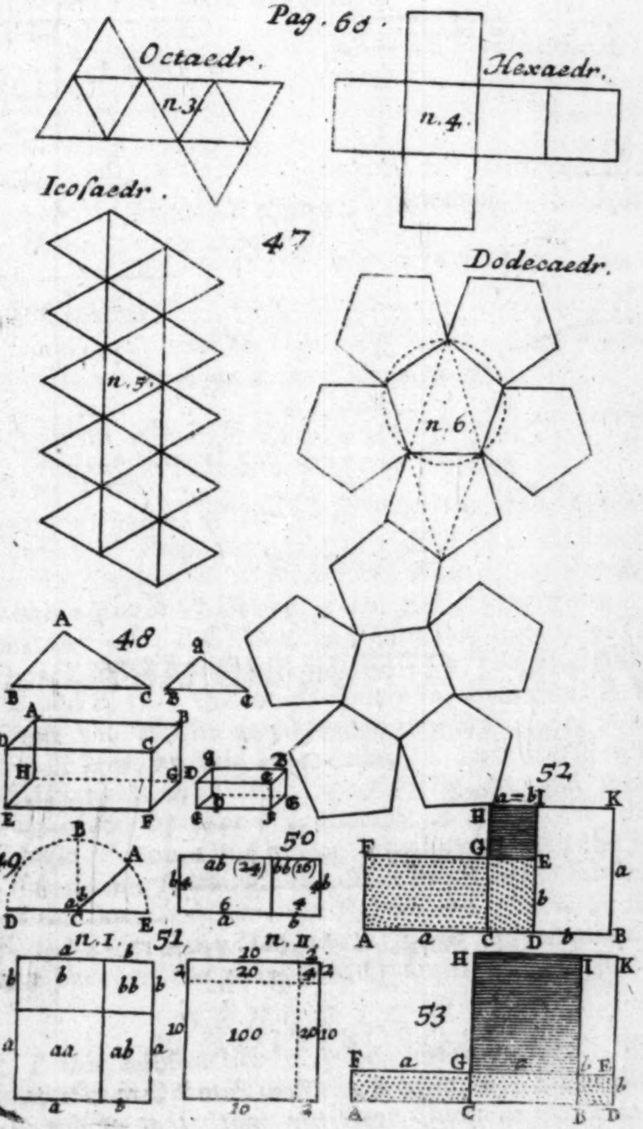
PROPOSITION X.

IF *a Quantity be divided any how into* * *two Parts, the Square of the Whole, together with that of one of its Parts, is equal to two Rectangles contained under the Whole and the firſt Part, together with the Square of the other Part.*

* Eucl. & Clav. Prop. 7. l. 2.

An

Pag. 60.



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An Univerſal Demonſtration.

Let a be one Part, and b the other,	
$a+b$ the Whole	The Whole $a+b$
a the firſt Part *	$a+b$
$aa+ab$	$aa+ab$
2	$ab+bb$
$2aa+2ab$ double Rectang.	□ of the whole $aa+2ab+bb$
add bb the □ of the other Part	* add aa
Sum $2aa+2ab+bb$ = to the Sum	$2aa+2ab+bb$
(*Vid. Fig.* 54. N° 1.)	Q. E. D.

CONSECTARY.

HENCE you have the *Subſtraction* of Surd Numbers, or more generally of Surd Quantities, by help of the following Rule.

Add the Squares of the given Roots together, (according to Conſect. 3. Prop. 7.*) and from their Sum ſubſtract the double Rectangle of their Roots; the Root of the Remainder will be the ſought Difference of the given Quantities.* p.58

As if the $\surd 8$ (B C) is to be ſubſtracted from $\surd 50$ (A C) (*Fig.* 54. N° 2.) you muſt add 50. (*i. e.* the whole Square A D) and 8, (*i. e.* the other Square ſuperadded D E,) and the Sum 58, will be equal to the two Rectangles A F and F E + □ G H, by this Propoſition. I find therefore thoſe two Rectangles by multiplying $\surd 50$ by $\surd 8$, and then the Product $\surd 400$ by 2 or $\surd 4$, thereby to obtain the double Rectangle $\surd 1600$, *i. e.* (having actually extracted the Root) 40. This double Rectangle therefore or 40, being ſubſtracted out of the ſuperiour Sum, the Remainder 18 will be the □ G H, and ſo its Root (*viz.* $\surd 18$.) gives the required Difference between the given Surd Quantities. p.80

SCHOLIUM.

BUT this Subſtraction may be performed yet a ſhorter way, if each Quantity under its Radical Sign can be divided by ſome Square, ſo that the ſame Quotient may come out on both ſides; that is, if the Surd Quantities are *Communicants*, as *e. g.* $\surd 50$ (the Number 50 being divided by 25) is equal to $5\surd 2$, and $\surd 8$ to $2\surd 2$; for then the Numbers prefix'd to the Radical Sign being ſubſtracted from one another (*viz.* $2\sqrt{2}$ from $2\sqrt{2}$) you'll have immediately the Remainder or Difference $3\sqrt{3}$, *i. e.* $\sqrt{18}$. But if the propoſed Quantities are not Communicants, (as if the $\sqrt{3}$ is to be ſubſtracted from $\sqrt{7}$) the Remainder may be briefly expreſſed by means of the Sign—thus, $\sqrt{7}-\sqrt{3}$, or according to the foregoing *Conſectary*, thus, $\sqrt{10-\sqrt{84}}$.

PROPOSITION XI.

IF any Quantity be divided into two Parts, (*a*) *the Quadruple Rectangle contained under the Whole and one of its Parts, together with the Square of its other Part, will be equal to the Square of the Quantity compounded of the Whole and the other Part.*

(*a*) Eucl. & Clav. viii.

Demonſtration.

Suppoſe $a+b$ the Whole.
b one Part.

$ab+bb$ the Rectangle of theſe two.
mult. by 4

$4ab+4bb$ the Quadruple Rectangle.
Add aa the Square of the other Part.

Sum $aa+4ab+bb$

The Quantity compounded of the Whole and the firſt Part $\Big\}$ $a+2b$
$a+2b$

$aa+2ba$
$2ba+4bb$

Square of the Compound Quantity $aa+4ba+4bb=$ to the former Sum.

(*Vid. Fig.* 55.) Q. E. D.

p. 80

PROPOSITION XII.

IF any Quantity be divided into two equal Parts (b) *and into two other unequal ones, the Squares of the unequal Parts taken together will be double the Square of half the Quantity, and the Square of the Difference,* viz. *of the equal and unequal Parts taken together.*

(b) Eucl. & Clav. Prop. 9

De-

Demonſtration.

Suppoſe the equal Parts to be a and a, the Difference b, and the greater of the unequal Parts $a+b$, and the leſs $a-b$.

The greater Part	The leſs	Half.	Differ.
$a+b$	$a-b$	a	b
$a+b$	$a-b$	a	b
□ $aa+2ab+bb$	□ $aa-2ab+bb$	□ aa	□ bb
The Sum of theſe $2aa+2abb$		Sum $aa+bb$	
Q. E. D.			

p 80

(*Vid. Fig.* 56.)

PROPOSITION XIII.

If to any whole Quantity (*) divided into two equal Parts there be added another Quantity of the ſame kind, the Square of the Quantity compounded of the Whole, and the Quantity added, together with the Square of the Quantity added, will be double the Square of half the Quantity, and the Square of the Sum of the half and the Part added taken together.

(*) *Euclid.* & *Clav. Prop.* 10.

Demonſtration.

Suppoſe the Whole to be $2a$, the half Parts a and a, and the Quantity added b; then the Quantity compounded of the Whole and the Quantity added, will be $2a+b$, and that of the half and the Quantity added $a+b$.

Qu. comp. of the Whole and Quantity added	Half	Qu. comp. of the half and qu. added,
$2a+b$	a	$a+b$
$2a+b$	a	$a+b$
$4aa+2ab$		
$2ab+bb$	□ aa	□ $aa+2ab+bb$
□ $4aa+4ab+bb$		Sum $2aa+2ab+bb$
□ Qu. added, bb		Manifeſtly the half of the former Sum. Q. E. D.
Sum $4aa+4ab+2bb$		

CHAP.

CHAP. III.

Of Progreſſion, or Arithmetical Proportionals.

PROPOSITION XIV.

IF there are three Quantities in continued Progreſſion, or in Arithmetical continued Proportion, the Sum of the Extremes is the double of the middle Term.

Demonſtration.

Such are *e.g.* $a+x$, $a+2x$ aſcending.
or a, $a-x$, $a-2x$ deſcending.

By *Definition* 32. the Sum of the Extremes in the firſt is $2a+2x$, in the latter $2a-2x$; in both manifeſtly double of the middle Term. Q. E. D.

PROPOSITION XV.

IF there are four of theſe continued Proportionals, the Sum of the Extreme Terms is equal to the Sum of the mean Terms.

Demonſtration.

Such are *e.g.* a, $a+x$, $a+2x$, $a+3x$, &c. aſcending, or a, $a-x$, $a-2x$, $a-3x$, &c. deſcending; in the one, the Sum of the Extremes is $2a+3x$, in the other $2a-3x$; and alſo of the Means $2a+3x$ and $2a-3x$. Q. E. D.

PROPOSITION XVI.

IF there are never ſo many of theſe continued Proportionals, the Sum of the Extremes is always equal to the Sum of any two others of them, equally remote from the Extremes, or alſo double of the middle Term, if the Number of the Terms be odd.

Demonſtration.

Suppoſe a, $a+x$, $a+2x$, $a+3x$, $a+4x$, $a+5x$, $a+6x$, *&c.* or a, $a-x$, $a-2x$, $a-3x$, $a-4x$, $a-5x$, $a-6x$, to be in Arithmetical Progreſſion; the Sum of the Extremes, as alſo of any two of them equally remote from the Extremes, and the double of the middle Term, in the firſt Series is $2a+6x$, in the latter, $2a-6x$, *&c.* Q. E. D.

SCHOLIUM. I.

NOR can we doubt but that this will always be ſo, how far ſoever the Progreſſion be continued; if you conſider that the laſt Term contains the firſt, and the common Difference ſo many times taken as is the number of Terms excepting one, but the firſt has no Difference added to it; and therefore ſince the laſt Term but one contains one Difference leſs than the laſt, and the ſecond, on the contrary, has one more than the firſt; conſequently the Sum of the one will neceſſarily be equal to the Sum of the other: and in like manner the laſt except two, contains two Differences leſs than the laſt; but, on the contrary, the third exceeds the firſt by a double Difference, the double Difference being added to it, *&c.* as is obvious to the Eye in our firſt univerſal Example. Hence you have theſe

CONSECTARIES.

I. YOU may obtain the Sum of any Terms in Arithmetical Proportion, if the Sum of the Extremes be multiplied by half the number of Terms, or (which is the ſame thing) half the Sum by the number of Terms.

II. To obtain therefore the Sum of 600, or never ſo many ſuch Terms, you need only have the Extremes and the Number of Terms: So that you have a very compendious Way of proceeding in Queſtions that are ſolvible by theſe Progreſſions, if, having the firſt Term and Difference of the Progreſſion given, you can obtain the laſt, neglecting the intermediate ones.

III. But you may obtain the laſt Term, by multiplying the given Difference by the given Number of Terms leſſened by Unity, and then adding the firſt Term to the Product; as is evident from the preceding *Scholium*.

IV. Hence we may eaſily deduce this Theorem, *viz.* that the Sum of any Arithmetical Progreſſion beginning from o, is ſubduple or $\frac{1}{2}$ of the Sum of ſo many Terms, equal to the greateſt, as is the number of Terms of that Progreſſion. For if the firſt Term be o, and the laſt x, and the given number of Terms a, the Sum of the Progreſſion will be $\frac{1}{2}ax$, by *Conſect.* 1. but the Sum of ſo many Terms equal to the greateſt is ax. Q. E D

SCHOLIUM II.

NOW if any one would be ſatisfied of the Truth of this laſt *Conſect.* without the literal or ſpecious Notes, let him conſider, that if the firſt Term be ſuppoſed to be *o*, the laſt (whatever it is) will be the Sum of the Extremes. The laſt therefore multiplied by half the number of Terms, gives the Sum of the Progreſſion, by *Conſect.* 1. and the ſame laſt Term multiplied by the whole number of Terms, gives the Sum of ſo many Terms equal to the greateſt. But that this muſt needs be double of the precedent is evident, becauſe any Multiplicand being multiplied by a double Multiplier, muſt needs give a double Product. Now as this Conſectary will be of ſingular Uſe to us hereafter for demonſtrating ſeveral Propoſitions, ſo the three former are the very ſame practical Rules of Arithmetick, which are commonly made uſe of in Arithmetical Progreſſions: for the Illuſtration whereof, *Swenterus* gives us ſeveral ingenious Examples in his *Delic.* Part I. Queſt. 70, &c.

CHAP. IV.

Of Geometrical Proportion in general.

PROPOSITION XVII.

(a) Euclid. 17. l. 6. & 20. l. 7.

IF there are three Quantities continually (a) *proportional, the Rectangle of the extreme Terms is equal to the Square of the mean Term.*

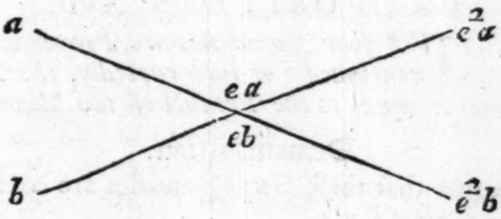
Demonſtration.

Such are e. g. a, ea, e^2a,	The mean Term, ea
The Extremes e^2a	ea
a	
Rectangle e^2a^2	Square e^2a^2 Q. E. D.

SCHO-

SCHOLIUM.

Moreover, if three Quantities on each ſide are in the ſame continual Proportion, as



the Rectangles of the Extremes made croſs-ways are equal to the Rectangle of the mean Terms; being both ways e^2ab.

Whence by the way may appear that Propoſition of *Archimedes*, * *That the Surface of a Right Cone is equal to a Circle, whoſe Radius is a mean Proportional between the Side of that Cone and the Semidiameter of the Baſe.* For ſuppoſe EF to be a mean Proportional between the Side of the Cone BC (*Fig.* 57.) and the Semidiameter of the Baſe CD, ſince an equal number of Peripheries anſwer to an equal Number of Radii in the ſame Proportion; then $\frac{1}{2} e^2ab$ half the Product of the firſt Line BC into the laſt Periphery *a*, (that is, by *Conſect.* 4. *Definit.* 18. the Surface of the given Cone) will be equal to $\frac{1}{2} e^2ab$ half the Product of the mean Line EF into the mean Periphery, (*i.e.* by *Conſect.* 2. *Definit.* 15.) to the Area of the Circle whoſe Radius is the mean Proportional EF. Q.E.D. p. 80

* *Archim. lib.* 1. *de Sphær.* & *Cyl. prop.* 14.

The ſame Propoſition of *Archimedes* may alſo be demonſtrated after this way: If the Side of the Cone BC be called *b*, and the Semidiameter of the Baſe *a* (that the Periphery, by *Conſect.* 1. *Definit.* 31. may be $2ea$, and ſo the Surface of the Cone, by *Conſect.* 4. *Definit.* 18. eab) then $\sqrt{ab}$ will be a mean Proportional between *b* and *a*, by this 17th *Propoſition*; which being taken for the *Radius*, the whole Diameter will be $2\sqrt{ab}$, and the Periphery $2\sqrt{ab}$; therefore by *Conſect.* 2. *Definit.* 15. half the Radius ($\frac{1}{2}\sqrt{ab}$) multiply'd by the Periphery (ſince $\sqrt{ab}$ multiplied by $\sqrt{ab}$, neceſſarily produces ab) will give you eab the Area of the Circle whoſe Radius is that mean † Proportional, equal to the Sufrace of the given Cone, which before was expreſſed in the ſame Terms. Q.E.D.

p. 49
- 29
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† *Archim. lib.* 1. *de Sphær.* & *Cyl. prop.* 15.

Hence alſo naturally flows this other Propoſition, *viz.* That the Surface of the Cone ($\frac{1}{2}e^2ab$) is to its Baſe ($\frac{1}{2}ab$) as the Side of the Cone ($e^2 b$) is to b the Radius of the Baſe, as appears from the Terms.

PROPOSITION XVIII.

* Eucl. 16. l. 6. & 19. l. 7.

IF * *four Quantities are Proportional, either continuedly or interruptedly, the Product of the Extremes is equal to the Product of the Means.*

Demonſtration.

Suppoſe firſt that theſe four Quantities are continual Proportionals, a, ea, e^2a, e^3a.

$$\begin{array}{r|r} \text{Extremes } e^3a & \text{Means } e^2a \\ a & ea \\ \hline \text{Prod. } e^3aa = & \text{Prod. } e^3aa. \end{array} \qquad \text{Q.E.D.}$$

Let, (2.) theſe four Quantities be Proportionals interrupted, a, ea, b, eb.

$$\begin{array}{r|r} \text{Extremes } eb & \text{Means } ea \\ a & b \\ \hline \text{Prod. } eba & \text{Prod. } eab. \end{array} \qquad \text{Q.E:D.}$$

SCHOLIUM.

ON this Theorem is founded the *Rule of Three* in Arithmetick; ſo called becauſe having three Numbers given, (2. 5. 8.) it finds an unknown fourth Proportional one. For although this fourth Number, as we have ſaid, be unknown, yet its Product by the firſt, *viz.* 2. is known; becauſe the ſame is equal to the Product of the Means 5 and 8. Wherefore the Rule directs to multiply the third by the ſecond, that you may thereby obtain the Product of the Extremes; which divided by one of the Extremes, *viz.* the firſt, neceſſarily gives the other, *i. e.* the fourth ſought.

PROPOSITION XIX.

Contrariwiſe, if two Products ariſing from the Multiplication of two Quantities, be equal; thoſe four Quantities will be at leaſt interruptedly Proportional.

Demonſtration.

Let eba, the Product of the Extremes, be equal to eab of the Means; then the Extremes will either be eb and a, or e and ba, or b and ea, as alſo the Means. But what way ſoever either is taken, there can be no other Diſpoſition or placing of them, than one of the following:

1. $eb : eb :: a : a$
$e :: ab$
$ea :: b$; or inverſly.
$a :: eb$
$ab :: c$
$b :: ea$
2. $e : e :: ba : ba$
$eb :: a$
$ea :: b$; or inverſly.
$ba :: e$
$a :: eb$
$b :: ea$
3. $b : b :: ea : ea$
$a :: cb$
$ba :: e$; or inverſly.
$ea :: b$
$eb :: a$
$e :: ba$; or inverting the Order of them all.

In all theſe Diſpoſitions there may be immediately ſeen a Geometrical Proportion, by what we have ſaid in *Definition* 31 and 33.

CONSECTARIES.

I. AS we have ſhewn one Sign of Proportionality in the Definition of it, *viz.* That the ſame Quotient will ariſe by dividing the Conſequents by the Antecedents; ſo now we have another Sign of it, *viz.* The Equality of the Products of the Extremes and Means.

II. By a bare Subſumption the Truth of *Prop.* 14. *lib.* 6. *Euclid.* may hence appear, at leaſt partly: Which we ſhall yet more commodiouſly ſhew hereafter.

PROPOSITION XX.

IF there are ever ſo many Continual Proportionals, the Product of the Extremes is equal to the Product of any two of the Means that are equally diſtant from the Extremes, as alſo to the Square of the mean or middle Term, if the Terms are odd.

Demonſtration.

Such are, *e.g.* a, ea, e^2a, e^3a, e^4a, e^5a, e^6a, &c. and the Product of the Extremes, and of any two Terms equally remote from them, and the Square of the mean or middle Term, every where, is e^6aa. Q.E.D.

SCHOLIUM I.

NOR can there be any doubt but this will always be ſo, how far ſoever the Progreſſion is continued; if you conſider that the laſt Term always contains the firſt multiplied by the name of the Ratio drawn ſo many times into itſelf, as is the Place of that Term in the Rank of Terms, excepting one; but the firſt Term is not multiplied at all by the name of the Ratio. Altho' therefore the laſt Term but one is in one (name or) degree of Ratio leſs than the laſt, the ſecond, on the contrary, is in one more than the firſt; therefore the Product of the one will neceſſarily be equal to the Product of the other. Thus alſo the laſt Term but two is two Degrees of Proportion lower than the laſt; but then the third being to be multiplied into that, exceeds the firſt by two Degrees of Proportion, *&c.* as may be ſeen in our Univerſal Example. Hence you have the following

CONSECTARIES.

I. HAving ſome of the Terms given in a continual Proportion (*e.g.* ſuppoſe 10.) you may eaſily find any other that ſhall be required (*e.g.* the 17th) as being an Extreme; if the 2 Terms given, being equally remote from the firſt and that required (as are *e.g.* the eighth and tenth) be multiplied by one another, and this Product, like that alſo of the Extremes, be divided by the firſt.

II. But this may be performed eaſier, if you moreover take in this Obſervation, *viz.* That if ever ſo many places of Proportionals, paſſing over the firſt, be noted or marked by Ordinals or Numbers according to their Places (as in this Univerſal Example)

$$\begin{array}{ccccccc} a, & ea, & e^2a, & e^3a, & e^4a, & e^5a, & e^6a, \\ & \text{I.} & \text{II.} & \text{III.} & \text{IV.} & \text{V.} & \text{VI.} \end{array}$$

The Place of the 7th Term is (*e.g.*) VI. (and ſo the Place of any other of them is leſs by Unity than its Number is among the Terms) and it may be conſidered as compoſed of the Places of any others, equally diſtant from the Extremes, *e.g.* V. and I. IV. and II. or twice III. *&c.*

III. Here you have the Foundation of the Logarithms, *i. e.* of a compendious way of Arithmetick, never enough to be praiſed. For if, *e. g.* a Rank of Numbers from Unity, continually proportional, be ſigned or noted with their Ordinals (as we have ſaid) as Logarithms,

1.	2.	4.	8.	16.	32.	64.	128.	256,	&c.
	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	

and any two of them (as 8 and 32) are to be multiplied together; add their Logarithms III. and V. and their Sum VIII. gives you the Logarithm of their Product 256; as the Terms equally remote from the 2 given ones and the firſt, and ſo whoſe Product into the firſt (which is Unity) *i. e.* into itſelf, will be equal to the Product of the Numbers to be multiplied: And contrariwiſe, if, *e. g.* 128 is to be divided by 4, ſubſtracting the Logarithm of the firſt II. from the Logarithm of the ſecond VII. the remaining Logarithm V. points out the Number ſought 32. So that after this way the Multiplication of Proportionals is, by a wonderful Compendium, turned into Addition, and their Diviſion into Subſtraction, and the Extraction of the Square Root into Biſecting or Halving, (for the Logarithm of the Square Number 16 being biſected, the half II. gives the Root ſought 4) and the Extraction of the Cube Root, into Triſection, (for the Logarithm of the Cube 64 being triſected, the third part gives the Cubick Root ſought 4.

SCHOLIUM II.

THAT we may exhibit the whole Reaſon of this admirable Artifice (which about 35 Years ago was found out by the Honourable Lord *John Neper* Baron of *Merchiſton* in *Scotland*, and publiſhed ſomething difficult, but afterwards rendered much eaſier and brought to perfection by *Henry Briggs*, the firſt *Savilian* Profeſſor of Geometry at *Oxford*:) I ſay, that we may exhibit the whole Reaſon of it in a Synopſis, after an eaſy Way, when its Uſe appeared ſo very conſiderable in great Numbers in the Tables of Sines and Tangents, nor yet could they be uſeful without mixing vulgar Numbers with them, eſpecially in the Practical Parts of Geometry, the buſineſs was to accommodate this Logarithmical Artifice to them both. Firſt therefore, that Artiſts may aſſign Logarithms to all the common Numbers proceeding from 1 to 1000 or 10000, &c. they firſt of all pick out thoſe which proceed in continued Geometrical Proportion, and particularly, tho' arbitrariouſly, thoſe which increaſe in a Decuple Proportion, *e. g.* 1. 10. 100. 1000. 10000, *&c.*

Under which muſt be written a Series of Ordinals in Arithmetical Progreſſion, according to *ConſeƐt.* 3. not ſubſtituting the ſimple Numbers, 1, 2, 3, *&c.* but thoſe augment with ſeveral Cyphers after them, that ſo we may alſo aſſign in whole Numbers Logarithms to the intermediate Numbers between 1 and 10, 10 and 100, *&c.* Wherefore by this firſt Suppoſition, Logarithms in Arithmetical Proportion, anſwer to Numbers in Geometrical Proportion, after the way we here ſee,

	1	10	100
Log.	0000000	10000000	20000000
		1000	10000
		30000000	40000000, *&c.*

As that they alſo exhibit certain Characteriſtical initial Figures, whereby you may ſee, that all the Logarithms between 1 and 10 begin from 0, the reſt between 10 and 100, from 1, the next from 100 to 1000 from 2, *&c.*

The Logarithms of the Primary Proportional Numbers being thus found, there remain'd the Logarithms of the intermediate Numbers between theſe to be found: For the making of which, after different ways, ſeveral Rules might be given, drawn from the Nature of Logarithms, and already ſhewn in *ConſeƐt.* 3. See *Briggs*'s *Arithmetica Logarithmica*, and *Gellibrand*'s *Trigonometria Britannica*; the firſt whereof, *chap.* 5. and afterwards, ſhews at length both ways delivered by *Neper* in his *Appendix*. But the buſineſs is done more ſimply by *A. Ulacq*, in his Tables of Sines, *&c.* whoſe mind we will yet further explain thus: If you are to find *e. g.* the Logarithm of the Number 9, between 1 and 10, augmented by as many Cyphers as you added to the Logarithm of 10, or the reſt of the Proportionals (*h. e.* between 10000000 and 100000000) you muſt find a Geometrical Mean Proportional, *viz.* by multiplying theſe Numbers together, and extracting the Square Root out of the Product 1000000000000000, by *Prop.* 17. Now if this Mean Proportional be leſs than 9 augmented by ſo many Cyphers; between it and the former Denary Number you muſt find a ſecond mean Proportional, then between this and that ſame a third; and ſo a fourth, *&c.* but if it be greater, then you muſt find a mean Proportional between it and the next leſs, *&c.* till at length, after ſeveral Operations, you obtain the number 99999998, approaching nearly to 90000000. Now if between the Logarithm of Unity and Ten (*i. e.* between 0 and 10000000) you take an Arithmetical Mean Proportional (05000000) by Biſecting their Sum by Prop. 14. and then between this and the ſame Logarithm of Ten, you take another Mean, and ſo a third, and a fourth, *&c.* at length you will obtain that which anſwers to the laſt abovemention'd, *viz.* 9. See the following Specimen.

A TABLE of the Geometrical Proportionals between 1 *and* 10, *augmented by* 7 *Cyphers, and of the Arithmetical Proportionals between* 0 *and* 10000000, *being the Logarithms correſponding to them.*

Geometrical Mean Proportionals.		*Arithmetical Mean Proportionals. Logar.*
31622777	Firſt,	05000000
56234132	Second,	07500000
74989426	Third,	08750000
86596435	Fourth,	09375000
93057205	Fifth,	09687500
89768698	Sixth,	09531250
91398327	Seventh,	09609375
90579847	Eighth,	09570312
90173360	Ninth,	09550781
89970801	Tenth.	09541015

Which is thus made: In the firſt Column a Geometrical Mean Proportional between 10 000 000 and 100 000 000 is the firſt Number of it; then another Mean between that and the ſame laſt Number 100000000, gives the ſecond; and ſo to the fifth, 93057205. Which, ſince it is greater than 90000000, another Mean between it and the precedent fourth Number becomes in order a fixth, but ſenſibly leſs than 90000000. Therefore between it and the fifth you will have a ſeventh Mean yet greater than 90000000; and between the ſixth and ſeventh, an eighth, ſomewhat nearer to 90000000, but not yet ſenſibly equal, but ſomewhat bigger: moreover between the ſixth and eighth you will have a ninth, between the ninth and ſixth a tenth, gradually approaching nearer to 90000000, but yet ſomewhat ſenſibly differing from it. Now if you continue this inquiry of a mean Proportional between this tenth, as ſomewhat too little, and the precedent ninth, as ſomewhat too big, and ſo onwards, you will at length obtain the Number 89999998, only differing two in the laſt place from nine augmented with ſeven Cyphers, or 90000000. But for the Logarithm of this in the ſecond Column, by the ſame proceſs you are to find Arithmetical Mean Proportionals between every two Logarithms anſwering to every two of the ſuperiour ones, till you find, *e. g.* the Logarithm of the tenth Number 09541015, and ſo at length the Logarithm of the laſt, not ſenſibly differing from 09542425, the Logarithm of 9.

Thus with a great deal of labour, but alſo with a great deal of advantage to thoſe that make uſe of them, having found the Logarithms of ſome of the Numbers between 1 and 10 and 10 and 100, *&c.* you may find innumerable ones of the other intermediate Numbers with much leſs labour, *viz.* by the help of ſome Rules, which may be thus obtain'd from *Conſect.* 3. of the precedent Propoſition. 1. *The Sum of the Logarithms of the Number Multiplying and the Multiplicand, gives the Logarithm of the Product.* 2. *The Logarithm of the Diviſor ſubſtracted from the Logarithm of the Dividend, leaves the Logarithm of the Quotient.* 3. *The Logarithm of any Number doubled, is the Logarithm of the Square; tripled of the Cube,* &c. 4. *The half Logarithm of any Number is the Logarithm of the Square Root of that Number; the third part of it, of the Cube Root, &c.* Thus, *e. g.* if you have found the Logarithm of the number 9, after the way we have ſhewn, by the ſame reaſon you may find the Logarithm of the number 5 (*viz.* by finding Mean Proportionals between the ſecond and the firſt number of our Table, and between their Logarithms, *&c.*) and by means of theſe two Logarithms you may obtain ſeveral others: Firſt, ſince 10 divided by 5 gives 2; if the Logarithm of 5 be ſubſtracted from the Logarithm of 10, you'll have the Logarithm of 2, by Rule the ſecond. Secondly, ſince 10 multiplied by 2 makes 20, and by 9 makes 90, by adding the Logarithms of 10 and 2, and 10 and 9, you'll have the Logarithms of the numbers 90 and 20, by Rule 1. Thirdly, ſince 9 is a Square, and its Root 3, half the Logarithm of 9 gives the Logarithm of 3, by Rule 4; ſince 90 divided by 3 gives 30, the Logarithm of this number may be had by ſubſtracting the Logarithm of 3 from the Logarithm of 90, by Rule the ſecond. Fifthly, 5 and 9 ſquared make 25 and 81, the Logarithms of 5 and 9 doubled give the Logarithms of theſe numbers by Rule 3. In like manner, Sixthly, the Sum of the Logarithms of 2 and 3, or the Difference of the Logarithms of 5 and 30, give the Logarithm of 6, and the Sum of the Logarithms of 3 and 6, or 2 and 9, gives the Logarithm of 18; the Logarithm of 6 doubled, gives the Logarithm of 36, &c. And after this way you may find and reduce into Tables, the Logarithms of Vulgar Numbers from 1 to 10000 (as in the Tables of *Strauch.* p. 182, &c.) or to 100000 (as in the *Chiliads* of *Briggs*) But as to the manner of deducing the Tables of Sines and Tangents from theſe Logarithms of Vulgar Numbers, we will ſhew it in *Schol.* of *Prop.* 55. only hinting this one thing p.134. before-hand; that this Artifice of making Logarithms is elegantly ſet forth by *Pardie* in his *Elements of Geometry*, p. 112. by a certain Curve Line thence called the *Logarithmical Line*; by the help whereof he ſuppoſes Logarithms may be eaſily made: and having found thoſe of the numbers between 1000 and 10000, he ſhews, that all others may be eaſily had between 1 and 1000. Whereof we ſhall diſcourſe more largely in *Schol. Definit.* 15. *lib.* 2. 149.

PROPOSITION XXI.

If the firſt Term of never ſo many Continual Proportionals be ſubſtracted from the laſt, and the Remainder divided by the name of the Ratio or Proportion leſſen'd by Unity, the Quotient will be equal to the Sum of all the Terms except the laſt.

Demonſtration.

ea

e^2a

e^3a

e^4a

e^5a

The laſt Term leſs the firſt e^6a-a

Divided by the name of the *Ratio* leſſen'd by Unity, will give *

$e-1$
$e-1$
$e-1$
$e-1$
$e-1$
$e-1$

* *The Quot.* $e^5a+e^4a+e^3a+e^2a+ea+a$; And it's evident from the Operation that the ſame will always happen, tho' the number of Terms be continued never ſo far.

CONSECTARIES.

I. WHerefore in adding never ſo great a Series of Geometrical Proportionals, ſince it is enough that the firſt and laſt Term, together with the name of the *Ratio* be known *by this Prop.* and having found at leaſt ſome of the Terms of the Proportion, any other may be afterwards found, whoſe place will be compounded of the places of the two p.70. antecedent ones, according to *Conſect.* 2. *Prop.* 20. *viz.* by Multiplying the Terms anſwering to the two above-mention'd places, and dividing the Product by the firſt Term; thence it will be very eaſy to add a great Series of Proportionals into one Sum, tho' the particular ſeparate Terms remain almoſt all of them unknown.

SCHOLIUM.

THeſe are the ſame Practical Arithmetical Rules concerning Geometrical Progreſſions; for the illuſtration of which, *Swenterus* in *Delic.* has given us ſo many pleaſant Examples, *lib.* 1. *Prop.* 59, *&c.* Firſt of all, that famous Example is of this kind, which relates to the Chequer-work'd Table or Board to fling Dice on, with its 64 little Squares, which Dr. *Wallis* has tranſlated out of the Arabick of *Ebn Chalecan*, into Latin, in *Oper. Mathem. Part.* 1. *Chap.* 31. for the illuſtration of which, we have heretofore compoſed an Exercitation, and ſhall here only note theſe few things: If there are ſuppoſed 64 Terms in a double Proportion from Unity, and the firſt of them, noted with their local Numbers, are theſe that follow;

1	2	4	8	16	32	64	128
	I	II	III	IV	V	VI	VII.

You may have the Term of the 13th place, *viz.* 8192, by multiplying together the VIth and VIIth place; and the Term of the XXVIth place, by ſquaring or multiplying this new Product again by itſelf; and moreover the Term of the LIId place, by multiplying that Product again by itſelf;

and furthermore the Term of the LIXth Place, by the Multiplication of the Number laſt found by the Number of the VIIth place; and laſtly the Term of the LXIIId Place (*i.e.* the laſt in the propoſed Series) by multiplying this laſt of all by the Number of the IVth Place.

II. Moreover you may by this Art collect infinite Series of Proportional Terms into one Sum, altho' it is impoſſible to run over all the Terms ſeparately, becauſe infinite, *e.g. in a continued Series of Fractions, decreaſing in a double Proportion* $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c. *ad infinitum*, if you take them backwards, you may juſtly reckon a Cypher or 0, for the firſt Term (for between $\frac{1}{2}$ and 0 there may be an infinite Number of ſuch Terms) *and the infinite Sum of theſe Terms will be preciſely equal to Unity*; for ſubſtracting the firſt 0 from the laſt $\frac{1}{2}$, and the Remainder $\frac{1}{2}$ being divided by the name of the Ratio leſſened by 1, *i.e.* by 1 which divides nothing, the Quotient $\frac{1}{2}$ is the Sum of all the Terms excepting the laſt, by *Prop.* 21; and ſo the laſt $\frac{1}{2}$ being added, the Sum of all in that Series will be 1. Now if the laſt be not $\frac{1}{2}$ but 1, the Sum of all will neceſſarily be 2; if 2 be the laſt, the Sum of all will be 4; in a word, it will be always double the laſt Term.

III. And ſince in this caſe the Sum of all the precedent Terms is equal to the laſt Term, the one being ſubſtracted from the other, there will remain nothing, *i. e.* $\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}-\frac{1}{32}$, *&c. in infinitum*, is $=0$, and alſo $1-\frac{1}{2}-\frac{1}{4}$, *&c.* or $2-1-\frac{1}{2}-\frac{1}{4}$, *&c.* $=0$.

IV. In like manner *the Sum of infinite Fractions decreaſing in a triple Ratio from* $+\frac{1}{3}$ $(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$, &c.) *will be equal to* $\frac{1}{2}$: for if from the laſt Term $\frac{1}{3}$ (again in an inverted Order) you ſubſtract the firſt 0, and the Remainder $\frac{1}{3}$ be divided by the Name of the Ratio leſſen'd by Unity, *i.e.* by 2, the Quotient $\frac{1}{6}$ will be the Sum of all the antecedent Terms; and adding to this laſt $\frac{1}{3}$ or $\frac{2}{6}$, the Sum of all will be $\frac{3}{6}$ or $\frac{1}{2}$.

V. Thus *an infinite Series of Fractions decreaſing from* $\frac{1}{4}$ *in a Quadruple Proportion* $(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$, &c.) *is equal to* $\frac{1}{3}$: for ſubſtracting the firſt 0 from the laſt $\frac{1}{4}$, and the Remainder $\frac{1}{4}$ being divided by the Name of the Proportion leſſened by Unity, *i.e.* by 3, you will have $\frac{1}{12}$ the Sum of all except the laſt; and adding alſo the laſt $\frac{1}{4}$ or $\frac{3}{12}$, you'll have the whole Sum $\frac{4}{12}$ or $\frac{1}{3}$.

VI. Thus alſo *an infinite Series of Fractions decreaſing* from $\frac{1}{5}$ *in a Quintuple Proportion* $(\frac{1}{5}+\frac{1}{25}+\frac{1}{125}$, *&c.*) is equal to $\frac{1}{6}$: $\frac{1}{6}+\frac{1}{36}+\frac{1}{216}$, &c. *is equal* to $\frac{1}{5}$, *&c.* and ſo any Series of this kind is equal to a Fraction whoſe Denominator is leſs by an Unit than the Denominator of the laſt Fraction in that Series.

VII. Generally alſo, *any infinite Series of Fractions decreaſing according to the Proportion of the* Denominator *of the laſt* Term, and having a common Numerator leſs by an Unit *than the Denominator of the laſt Term.* (*e.g.* $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}$, &c. *or* $\frac{3}{4}+\frac{3}{16}+\frac{3}{64}$, &c. *or* $\frac{4}{5}+\frac{4}{25}+\frac{4}{125}$, &c.) *is equal to*
p.77. *Unity*, after the ſame way as the Series of *Conſect.* 2. which may be comprehended under this kind, and which may be demonſtrated in all its particular Caſes by the ſame Method we have hitherto made uſe of, or alſo barely ſubſumed from
ib. *Conſect.* 4, 5, and 6. For ſince $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$, *&c.* is equal to $\frac{1}{2}$; $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}$ will be equal to $\frac{2}{2}$, or 1; and ſo on in the reſt.

VIII. Particularly *the Sum of* $\frac{3}{72}+\frac{3}{288}+\frac{3}{1152}$, &c. *decreaſing in a Quadruple Proportion, is equal to* $\frac{1}{18}$; and the Sum of $\frac{3}{64}+\frac{3}{256}+\frac{3}{1024}$, *&c.* is equal to $\frac{1}{16}$; and the Sum of $\frac{7}{64}+\frac{7}{512}+\frac{7}{4096}$, *&c.* decreaſing in an Octuple Proportion, is equal to $\frac{1}{8}$: For ſubſtracting the firſt Term 0, and dividing the Remainder by the Name of the Ratio leſſen'd by 1, *i.e.* by 3, the Quotient $\frac{1}{72}$ gives the Sum of all except the laſt. This therefore, (*viz.* $\frac{3}{72}$) being added, the Sum of all will be $\frac{4}{72}$, or $\frac{1}{18}$: In like manner $\frac{7}{64}$ being divided by the Name of the Ratio leſſened by Unity, the Quotient will give $\frac{1}{64}$; and adding the laſt, the Sum of all will be $\frac{8}{64}$, *i.e.* $\frac{1}{8}$. So that hence it is evident, that $\frac{1}{18}-\frac{3}{72}-\frac{3}{288}-\frac{3}{1152}$, *&c.* or $-\frac{1}{18}+\frac{3}{72}+\frac{3}{288}+\frac{3}{1152}$, *&c. in infinitum*, will be equal to nothing; alſo $\frac{1}{8}-\frac{7}{64}-\frac{7}{512}-\frac{7}{4096}$, *&c.* $=0$.

IX. *The Sum of a ſimple Arithmetical Progreſſion* (i.e. *aſcending by the Cardinal Numbers) continued from* 1, ad infinitum, *is ſubduple, or half of the Sum of the ſame Number of Terms, each of which is equal to the greateſt*; *or, on the contrary, this latter Sum is double of the former.*
65. We might have ſubſumed this in *Conſect.* 4. *Prop.* 16. for, prefixing a Cypher before Unity, it will be a Caſe of that *Conſectary*, the Sum of the Progreſſion remaining ſtill the ſame. But that this is true in an infinite Series beginning from Unity, (for in a finite or determinate one, the Proportion of the Sum is always leſs than double, tho' it always approaches to it, and comes ſo much the nearer by how much greater the Series is) we ſhall now thus demonſtrate: To the Sum of the three Terms, 1, 2, 3, *i.e.* 6. the Sum of as many equal in number to the greateſt, *i.e.* 9, has the

the ſame Proportion as 3 to 2; but to the Sum of ſix Terms, 1, 2, 3, 4, 5, 6, *i. e.* 21, the Sum of as many equal to the greateſt, *i. e.* 36, has the ſame Proportion as 3 to $1+\frac{3}{4}$, that is, as 3 to $2-\frac{1}{4}$, the decreaſe being $\frac{1}{4}$: but to the Sum of 12 Terms, which may be found (by *Conſect.* 1. *Prop.* 16) = 78, p. 65. the Sum of ſo many equal to the greateſt, *viz.* 144. has the ſame Proportion, (dividing both ſides by 48) as 3 to $1\frac{30}{48}$, *i. e.* as 3 to $1+\frac{1}{2}+\frac{1}{8}$ (for $\frac{24}{48}=\frac{1}{2}$, and the remainder $\frac{6}{48}$ is the ſame as $\frac{1}{8}$) that is, as 3 to $2-\frac{1}{4}-\frac{1}{8}$, the decrement being now $\frac{1}{8}$. Since therefore, by doubling the number of Terms onwards, you'll find the decrement to be $\frac{1}{16}$, and ſo onwards in double Proportion; the Sum of an infinite Number of ſuch Terms, in Arithmetical Progreſſion, equal to the greateſt, will be to the Sum of the Progreſſion from 1, *ad infinitum*, as 3 to $-2\frac{1}{4}-\frac{1}{8}-\frac{1}{16}$, *&c.* that is, by *Conſect.* 2 and 3, as 3 to 77. $2-\frac{1}{2}$, that is, as 3 to $1\frac{1}{2}$, or as 2 to 1. Q. E. D.

X. *The Sum of any Duplicate Arithmetical Progreſſion* (*i. e.* a Progreſſion of Squares of whole Numbers aſcending) *continued from* 1 ad Infinitum, *is ſubtriple or* $\frac{1}{3}$ *of the Sum of as many Terms equal to the greateſt as is the number of Terms:* For any ſuch finite Progreſſion is greater than the ſubtriple Proportion, but approaches nearer and nearer to it continually, by how much the farther the Series of the Progreſſion is carried on. Thus the Sum of 3 Terms 1, 4, 9 = 14 is to thrice 9 = 27 as $1\frac{5}{9}$, or $1\frac{10}{18}$, or $1+\frac{1}{2}+\frac{1}{18}$ to 3 (dividing both ſides by 9). The Sum of ſix Terms, 1, 4, 9, 16, 25, 36, *viz.* 91. to ſix times 36, *i. e.* to 216 (dividing both ſides by 72) is as $1+\frac{1}{4}+\frac{1}{72}$ to 3; and the Sum of 12 Terms 650, to 12 times 144, *i. e.* to 1728 (dividing both ſides by 576) is as $1+\frac{1}{8}+\frac{1}{288}$ to 3, *&c.* the Fractions adhering to them thus conſtantly decreaſing, ſome by their half parts, others by three quarters (for $\frac{1}{18}$ is $\frac{4}{72}$; therefore the firſt decrement is $\frac{3}{72}$, and $\frac{1}{72}$, is $\frac{4}{288}$; therefore the ſecond decrement is $\frac{3}{288}$, &c.) Wherefore the Sum of the Infinite Progreſſion will be to the Sum of the like number of Terms equal to the greateſt, as

$$\begin{array}{l} 1+\frac{1}{2}+\frac{1}{18} \\ -\frac{1}{4}-\frac{3}{72} \\ -\frac{1}{8}-\frac{3}{288} \\ -\frac{1}{16}, \&c. \frac{3}{1152}, \&c. \end{array}$$

to 3, that is, by *Conſect.* 3 and 8, as 1 to 3. Q. E. D. 77. 78.

XI. *The Sum of a triplicate Arithmetical Progreſſion* (i. e. *aſcending by the Cubes of the Cardinal Numbers*) *proceeding from* 1 *thro'* 27, 64, &c. ad Infinitum, *is Subquadruple*, or $\frac{1}{4}$ *of the Sum of the like number of Terms equal to the greateſt*. For the Sum of 4 Terms, 1, 8, 27, 64, *i e.* 100, to 4 times 64, *i. e.* 256 (dividing both ſides by 64) will be found to be as $1+\frac{1}{2}+\frac{1}{16}$ to 4; but the Sum of 8 Terms, 1, 8, 27, 64, 125, 216, 343, 512, *i. e.* 1296 to 8 times 512, that is 4096 (dividing both Sides by 1024) will be found to be as $1+\frac{1}{4}+\frac{1}{64}$ to 4, *&c.* The adhering Fractions thus conſtantly decreaſing, ſum by their $\frac{1}{2}$ parts, the others by $\frac{1}{4}$, for $\frac{1}{16}$ is $\frac{4}{64}$, and $\frac{1}{64}$ is $\frac{4}{256}$, &c. Wherefore the Sum of the Infinite Progreſſion will be to the Sum of a like (Infinite) number of Terms, equal to the greateſt, as

$$\begin{array}{llll} 1+\frac{1}{2} & & +\frac{1}{8} & \\ \quad -\frac{1}{4} & & -\frac{1}{64} & \\ \quad -\frac{1}{8}, & \&c. & -\frac{1}{256}, & \&c.\ \text{to}\ 4; \end{array}$$

p.77,78. that is, by *Conſect.* 3 and 8, as 1 to 4. Q. E. D.

XII. *The Sum of an Infinite Progreſſion, whoſe greateſt Term is a Square Number, and the others do decreaſe according to the odd numbers* 1, 3, 5, 7, &c. *is in a Subſeſquialteran Proportion to the Sum of the like number of equal Terms*, i. e. *as* 2 *to* 3. For the Sum of three ſuch Terms, *e. g.* 9, 8, 5, *i. e.* 22 to thrice 9, *i. e.* 27. is (dividing both ſides by 9) as $2\frac{4}{9}$, *viz.* $\frac{8}{18}$ to 3, or $2+\frac{1}{2}-\frac{1}{18}$ to 3. But the Sum of ſix ſuch Terms, 36, 35, 32, 27, 20, 11, *i. e.* 161, to ſix times 36, *i. e.* 216 (dividing both ſides by 72) is as $2+\frac{1}{4}-\frac{1}{72}$, *&c.* the adhering Fractions thus always decreaſing ſome by $\frac{1}{2}$, others by $\frac{1}{4}$, as above in *Conſect.* 10. Wherefore the Sum of the Infinite Progreſſion will be to the Sum of the like number of Terms equal to the greateſt, as 79.

$$\begin{array}{lll} 2+\frac{1}{2} & -\frac{1}{18} & \\ \quad -\frac{1}{4} & +\frac{1}{72} & \\ \quad -{}^{1} & +\frac{1}{288}, & \&c.\ \text{to}\ 3, \end{array}$$

i. e. by *Conſect.* 3 and 8, as 2 to 3. Q. E. D. 77,78.

SCHOLIUM II.

THus, after our Method, have we demonſtrated the chief Foundations of the *Arithmetick* of *Infinites*, firſt found out by Dr. *John Wallis*, Savilian Profeſſor of Geometry at *Oxford*, and afterwards carried further by *Dethlerus Cluverus*, and

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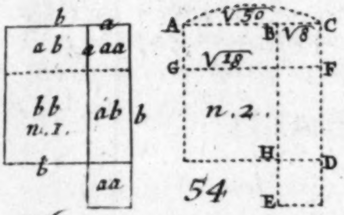

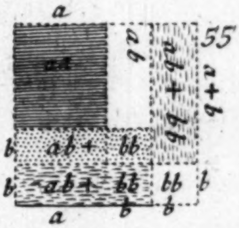

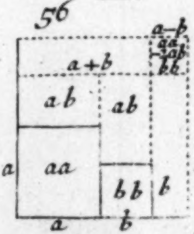

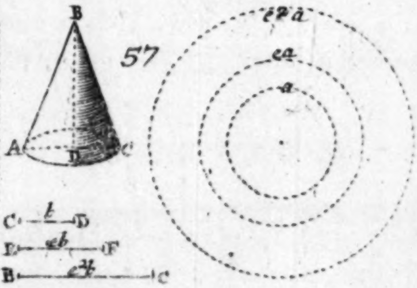

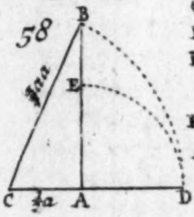

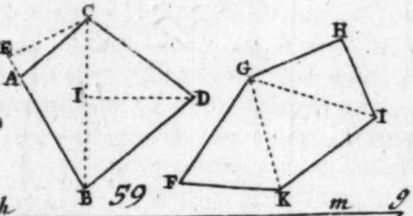

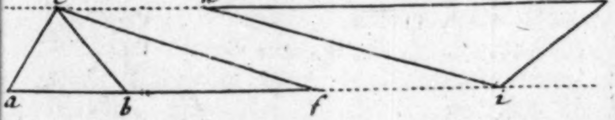

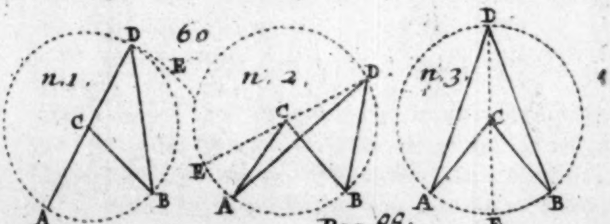

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and *Iſmael Bullialdus*. And from theſe Foundations in the following Treatiſe we will demonſtrate, directly and *a priori*, in a few Lines, the chief Propoſitions of Geometry, about which the Antients have ſpent ſo much labour, and compoſed ſuch large Volumes to demonſtrate, and that but indirectly neither.

PROPOSITION XXII.

THE Powers of Proportionals whether continuedly or diſcretely, ſuch as the Squares, Cubes,* &c. *are alſo Proportional.*

* *Euclid. lib.* 6. *prop.* 22.

An Ocular Demonſtration.

	Continual Proportionals.				Diſcrete Proportionals.			
	a	ea	e^2a	e^3a	a	ea	b	eb
Squares	aa	e^2a^2	e^4a^2	e^6a^2	a^2	e^2a^2	b^2	e^2b^2
Cubes	a^3	e^3a^3	e^6a^3	e^9a^3	a^3	e^3a^3	b^3	e^3b^3

Q. E. D.

SCHOLIUM.

IN this Truth is founded, 1. The Reaſon of the Multiplication and Diviſion of Surd Quantities: For ſince from the Nature and Definition of Multiplication, it is certain, that 1 is to the Multiplier as the Multiplicand to the Product, (for the Multiplicand being added as many times to it ſelf as there are Units in the Multiplier, makes the Product) if the $\sqrt{5}$ is to be multiplied by $\sqrt{3}$; then as 1 to the $\sqrt{3}$, ſo the $\sqrt{5}$ to the Product; and by the preſent Propoſition, as 1 to 3, ſo 5 to the □ of the Product, *i.e.* to 15. Wherefore the Product is $\sqrt{15}$; and ſo the Rule for multiplying Surd Quantities is this: *Multiply the Quantities under the Radical Signs, and prefix a Radical Sign to the Product.* Likewiſe ſince it is certain from the Nature of Diviſion, that the Diviſor is to the Dividend as 1 to the Quotient, (for the Quotient by its Units expreſſes how many times the Diviſor is contain'd in the Dividend) if the $\sqrt{15}$, is to be divided by $\sqrt{5}$, you'll have $\sqrt{5}$ to $\sqrt{15}$ as 1 to the Quotient; and, by the preſent *Prop.* 5 is to 15, as 1 to the □ of the Quotient, *i. e.* to 3. Therefore the Quotient is the Root of 3, and ſo the Rule for dividing Surd Quantities is this ; *viz. Divide the Quantities themſelves under the Radical Signs, and prefix the Radical Sign to the Quotient.*

II. Hence alſo flows the uſual Reduction of Surd Quantities to others partly Rational, and contrariwiſe, of thoſe to the form of Surds ; *e. g.* If you would reduce this mixt Quantity $2a\sqrt{b}$, *i. e.* $2a$ multiplied by the $\sqrt{b}$, to the form of a Surd Quantity ; which ſhall all be contain'd under a Radical Sign: The Square of the Rational Quantity, without the Sign $4aa$, if it be put under a Radical Sign, in this form $\sqrt{4aa}$, is equivalent to the Rational Quantity $2a$; but the $\sqrt{4aa}$ being p. 81. multiplied by $\sqrt{b}$ makes $\sqrt{4aab}$. by N°. 1. of this Scholium. Therefore $\sqrt{4aab}$ is alſo equivalent to the Quantity firſt propoſed $2a\sqrt{b}$. Contrariwiſe therefore, if this form of a meer Surd Quantity $\sqrt{4aab}$, is to be reduc'd to one more Simple, which may contain without the Radical Sign whatever is therein Rational ; by dividing the Quantiy comprehended under the ſign $\sqrt{}$ by ſome Square or Cube, &c. as here by $4aa$, (*i. e.* $\sqrt{4aab}$ by $\sqrt{4aa}$, i. e. $2a$) the Quotient will be $\sqrt{b}$, which multiplied by the Diviſor $2a$, will rightly expreſs the propoſed Quantity under this more ſimple Form $2a\sqrt{b}$: Which may alſo ſerve further to illuſtrate the Scholia 58.61 of *Prop.* 7. and 10.

PROPOSITION XXIII.

If there are four Quantities Proportional, (a, ea, b, eb) *they will be alſo Proportional.*

* *Eucl.* 15, 16, v. 9,10. 13, vii. 18. v.

* 17, v.

† *Eucl.* 1. 12. v. 5. 6. 12. vii.

1. Inverſly, *ea* to *a* as *eb* to *b*.
2. Alternatively, * *a* to *b* as *ea* to *eb*
3. Compoundedly, † $a+ea$ to ea, ſo $b+eb$ to eb.
4. Converſly, $a+ea$ to a as $b+eb$ to b.
5. Dividedly,* $a-ea$ to $\left(\begin{array}{l} ea \text{ as } b-eb \\ \text{or } a \end{array}\right.$ to $\left(\begin{array}{l} eb. \\ \text{or } b. \end{array}\right.$
6. † By a Syllepſis, a to ea as $a+b$ to $ea+eb$.
7. By a Dialepſis, a to ea as $a-b$ to $ea-eb$.

Which are all manifeſt, by comparing the Rectangles 69. of the Means and Extremes according to *Prop.* 19. and *Conſect.* 1. of it, or by dividing any of the Conſequents by 48. their Antecedents, according to *Def.* 31.

PROPOSITION XXIV.

IF in a double Rank of Quantities it is*
as a *to* ea, *and alſo* *as* ea *to* oa,
ſo b *to* eb, *ſo* eb *to* ob,
then by Proportion of Equality orderly placed, it will be alſo
as the firſt a, *to the laſt* oa, *in the firſt Series*;
ſo the firſt b, *to the laſt* ob, *in the ſecond Series.*
Which is manifeſt from the Terms themſelves.

* *Eucl.* 3. 20, 22. *lib.* v. & 14. *lib.* vii.

PROPOSITION XXV.

BUT if they are diſorderly placed,
† *as* oa *to* ea | *as* ea *to* a
ſo eob *to* ob | *ſo* ob *to* eb
you'll have here again by Proportion of Equality,
as the firſt oa *to the laſt* a, *in the firſt Series*;
ſo the firſt eob *to the laſt* eb, *in the ſecond Series.*
As is evident from the Rectangles of the Extremes and Means, as alſo from the very Terms.

† *Euclid.* 21, 23. *lib.* v.

PROPOSITION XXVI.

IF (*a*) *the Whole* ea *be to the Whole* a, *as the Part* eb *to the Part* b; *then alſo will*
the Remainder *Remainder* *Whole* *Whole.*
ea — eb *be to the* a — b, *as the* ea *to the* a
This is evident from the Rectangles of the Extremes and Means, both of which are eaa — eab. *Q. E. D.*

(*a*) *Euclid.* 5 & 19. *lib.* v. 7 & 11. *lib.* vii.

PROPOSITION XXVII.

REctangles or Products (*b*) *having one common Efficient or Side, are one to another as the other Efficients or Sides.*

(*b*) *Eucl.* 17, 18. *lib.* vii. *Beſides ſeveral others.*

Demonſtration.

Suppoſe the Products to be *ab* and *ac*, having the common Efficient *a*; I ſay it is

as *b* to *c*, ſo *ab* to *ac*.

Which is evident at firſt ſight, by comparing the Products of the Extremes and Means, and withal fully ſhows that other Way of proving Proportionality, whereby, by dividing the Conſequents by their Antecedents, the Identity or Sameneſs of the Quotients are wont to be demonſtrated.

SCHOLIUM I.

I. THE Reduction of Fractions, either to more compounded or more ſimple ones, is founded on this Theorem; on the one hand, by multiplying, and on the other, by dividing both the Numerator and Denominator by the ſame Quantity, as, *e.g.* $\frac{c}{b}$ and $\frac{ac}{ab}$ and $\frac{cac}{eab}$, $\frac{1}{3}$, $\frac{3}{9}$, $\frac{4}{12}$, &c. are in reality the ſame Fractions. And,

II. The Reduction of Fractions to the ſame Denominator, as if $\frac{b}{c}$ and $\frac{a}{d}$ are to be changed into two others that ſhall have the ſame Denominator; this is to be done by multiplying the Denominators together for a new Denominator, and each Numerator by the Denominator of the other for a new Numerator, and you'll have for the two Fractions above $\frac{bd}{cd}$ and $\frac{ac}{cd}$.

SCHOLIUM II.

TO conclude with Proportionals, we will here ſhew the Way of cutting or dividing any Quantity in *Mean and Extreme Ratio:* For the greater Part put x, the leſs will be $a-x$; and ſo by the *Hypoth.* theſe three Quantities a, x, p. 51. and $a-x$, will be proportional, by *Def.* 34. Therefore, 66. by *Prop.* 17. the Product of the Extremes $aa-ax=$ to the Square of the Mean xx, and (adding ax on both ſides) aa will be $=xx+ax$; and moreover, adding on both ſides $\frac{1}{4}aa$, you will have $\frac{5}{4}aa=xx+ax+\frac{1}{4}aa$. Now ſince this laſt Quantity is an exact Square, whoſe Root is $x+\frac{1}{2}a$, you will have $\sqrt{\frac{5}{4}aa}=x+\frac{1}{2}a$, and (ſubſtracting from both ſides $\frac{1}{2}a$) $\sqrt{\frac{5}{4}aa}-\frac{1}{2}a$ will be $=x$.

Now therefore we have a Rule to determine the greater Part of a given Quantity to be divided in Mean and Extreme Ratio; for if the given Quantity be a Line, *e.g.* $AB=a$ 80. (*Fig.* 58.) join to it † at Right Angles $AC=\frac{1}{2}a$: Wherefore, by the Theorem of *Pythagoras* 19. from *Schol. Definit.* 13. the Hypothenuſe CB, or CD (which is equal to it) is $=\sqrt{\frac{5}{4}aa}$; and conſequently $AC=\frac{1}{2}a$ being taken out of CD, the Remainder AD, or AE, which is equal to it, will be $=x$, the greateſt Part ſought, according

† Eucl. 11. lib. 2. & 30. lib. 6.

ding to *Euclid*, whoſe Invention this firſt Specimen of Analyſis, by way of Anticipation, reduces to its original Fountain. As for Numbers (tho' none accurately admits of this Diviſion) the Senſe of the Rule, or which is all one as to the thing itſelf, is this: *Add together the Squares of a whole Number and its half, and ſubſtract the ſaid half from the Root of the Sum.* (which can't be had exactly, ſince it is $\sqrt{\frac{5}{4}}$.)

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CHAP. V.

Of the Proportion or Ratio's of Magnitudes of the ſame kind in particular.

PROPOSITION XXVIII.

Triangles and Parallelograms, alſo Pyramids, and Priſms, and Parallelepipedons; laſtly, Cones and Cylinders, each kind compared among themſelves, if they have the ſame Altitude, are in the ſame Proportion to one another as their Baſes.

Demonſtration.

This and the following Propoſition, by a bare Subſumption, might have been added as *Conſectaries* to the precedent one; for the Altitudes in the one, and Baſes in the other, may be looked on as common Efficients, and the Magnitudes mentioned as their Products: But for the greater diſtinction ſake, we will thus demonſtrate them more particularly.

I. If the equal Altitudes of two Triangles, or * two Parallelograms, are called *b*, and the Baſe of the one *a*, and of the other *ea*; theſe Products will be *ba* and *bea*; and $\frac{1}{3}$ *ba* and $\frac{1}{2}$ *bea*, by *Def.* 28. *Schol.* 2. p. 43.

* *Eucl. Prop.* 1. *lib.* vi.

II. Likewiſe the equal Altitudes of two Priſms † or Pyramids, may be called *b*, and the Proportion of their Baſes expreſſed by *a* and *ea*; and the Priſms will be among themſelves as *ba* to *bea*, and the Pyramids as $\frac{1}{3}$ *ba* to $\frac{1}{2}$ *bea*, by the ſaid *Schol. Numb.* 3. ib.

† *Prop.* 5, 6. *lib.* xii. 25, 32. xi. & *Conſ.* 30 & 31, *of the ſame.*

III. There is alſo the ſame Proportion of Cylinders and Cones as of Pyramids and Priſms, by *Conſect.* 4. *Definit.* 17. But, 27.

Prop. 11. *lib.* xii.

as a to ea ſo is ba to bea.
and $\frac{1}{2}\,ba$ to $\frac{1}{2}\,bea$.
and $\frac{1}{3}\,ba$ to $\frac{1}{3}\,bea$. Q. E. D.

CONSECTARY.

* Prop. 35, 36, 37, 38, 39, 40. lib. I. & 29, 30, 31, lib. xi.

THerefore Magnitudes of the ſame kind upon the ſame or equal Baſes *, and of the ſame height, are equal among themſelves, and contrariwiſe.

PROPOSITION XXIX.

† Schol. prop. 1. lib. vi. 13, 14. l. xii.

TRiangles and Parallelograms, Pyramids and Priſms and Parallelepipedons, Cones and Cylinders, being on equal Baſes, are in the ſame Proportion as their Heights. †

Demonſtration.

Let all their Baſes be call'd a, and the Proportions of their Heights be as b to eb: Therefore, 1. the Parallelograms, Parallelepipedons and Cylinders, are one to the other of the ſame kind, as ba to eba; the Triangles as $\frac{1}{2}ba$ to $\frac{1}{2}eba$; the Pyramids and Cones as $\frac{1}{3}ba$ to $\frac{1}{3}eba$, by *Def.* 28. *Schol.* 2. p. 43. But,

as b to eb, ſo is ba to eba.
and $\frac{1}{2}ba$ to $\frac{1}{2}eba$.
and $\frac{1}{3}ba$ to $\frac{1}{3}eba$. Q. E. D.

PROPOSITION XXX.

* Lib. vi. prop. 14. 15. lib. xi. 34. lib. xii. 11, and its Coroll. alſo Prop. 15.

*EQual * Triangles, Parallelograms, Priſms, Parallelepipedons, alſo equal Pyramids, Cones and Cylinders have their Baſes and Heights reciprocally Proportional.*

Demonſtration.

For if for the equal Triangles you put $\frac{1}{2}\,a\,b$, for the Cones and Pyramids $\frac{1}{3}\,ab$, and for the reſt $a\,b$;

Whether the Baſes of the equal Quantities are ſuppoſed to be a, and ſo the Altitudes on both ſides b; or if the Baſe of the one be a and b the Altitude, but the Baſe of the other b and the Altitude a, you'll certainly have eitherways,

as a to a, ſo Reciprocally b to b;

the Baſe of the former to the Baſe of the latter, as the Altitude of the latter to the Altitude of the former, or,

as *a* to *b*, ſo Reciprocally *a* to *b*. Q. E. D.

CONSECTARY.

AND thoſe Magnitudes of the ſame kind, whoſe Baſes and Altitudes are thus Reciprocal, are equal by *Prop.* 18. p. 68.
for the Product or Rectangle of the Extremes is *a b*, and that of the Means *b a*.

PROPOSITION XXXI.

Triangles, Parallelograms, Priſms, Parellelepipedons, Pyramids, Cones and Cylinders, each kind compared among themſelves, are in the Proportion compounded of the Proportions of their Altitudes and Baſes. * * *Prop.* 23. *l.* vi.

Demonſtration.

Suppoſe the Baſe of the one to be *a*, and of the other *ea*, and the Altitude of the one *b*, of the other *i b*; therefore the one will be to the other,

as *a b* to *eiab*,
or $\frac{1}{2}$ *ab* to $\frac{1}{2}$ *eiab*,
or $\frac{1}{3}$ *ab* to $\frac{1}{3}$ *eiab*; *i. e.* every where as *a* to *eia*,
i. e. in the Proportion compounded of *a* to *ea*, and of *b* to
i b, by *ConſeCt.* 2. *Def.* 34. Q. E. D. 52

SCHOLIUM.

FRom what we have hitherto Demonſtrated, we may not only make an eſtimate of Magnitudes of the ſame kind compared together, which is eaſy to any one who attentively conſiders them; but alſo with *F. Morgues*, deduce a General Rule of expreſſing the Proportions of any Rectilinear Planes, or Solids contained under Plane Surfaces, by the proportion of one Right Line to another. For ſince Planes may be reſolved into Triangles, and the Solids into Pyramids having, (1.) two Rectilinear Planes given and thus reſolved, upon a Right Line, make the △ *abc* (*Fig.* 59.) equal to one 80.
of the Triangles of either of the Planes, *e. g.* to ABC; then having drawn the Parallel *cm*, if the △ BCD has the ſame Altitude with ABC, you need only join the Baſe BC to the Baſe *a b*. But if the Altitude DI is greater than the Alti-

tude

tude of the other, *e. g.* by $\frac{1}{5}$, then you muſt make *bf* equal to the Baſe BC augmented by a fifth part of itſelf, and the Triangle *bcf* will = BCD, and the whole *acf* = to the Rectilinear Figure ABCD. Now therefore if I likewiſe make another Triangle *ghi* equal to the other Rectilinear Figure FGHIK between the ſame Parallels; then will the Δacf be to the Δghi, that is, the Right-lined Figure ABCD to the Right-lined Figure FGHIK, as *af* to *gh*,
p. 85. by *Prop.* 28. 2. Having two Solids contained under Planes given, and having reſolved them into triangular Pyramids, they may be transferr'd between two parallel Planes, *viz.* by augmenting or diminiſhing their triangular Baſes reciprocally, according to the Exceſs or Defect of their Altitudes, as was done above with the linear Baſes; then thoſe triangular Baſes on both ſides, may be converted into one triangular Baſe, and conſequently each Solid into a Pyramid equal to itſelf; which two Pyramids will be one to the other as their triangular Baſes. And becauſe the Proportions of theſe Baſes may be reduced to the Proportion of two Lines each to the other, by N° 1. of this; thereſore alſo the *Ratio* or Proportion of the two Solids may be expreſſed by the Proportion of two Lines. Q. E. D.

PROPOSITION XXXII.

* Eucl. Prop. 2. l. xii.

CIRCLES * *are in the ſame Proportion to one another, as the Squares of their Diameters.*

Demonſtration.

Suppoſe *a* to be the Diameter of one Circle, and *b* of an-
49. other; then by *Definit.* 31. *Conſect.* 1. the Area of the one will be $\frac{1}{4}eaa$, and that of the other $\frac{1}{4}ebb$. But as *aa* is
69. to *bb*, ſo is $\frac{1}{4}eaa$ to $\frac{1}{4}ebb$, by *Conſect.* 1. *Prop.* 19. Q. E. D.

CONSECTARY I.

THE ſame will in like manner be manifeſt of ſimilar Sectors of Circles, while for the Parts of the Periphery you put *ia* and *ib*, like as for the Wholes we put *ea* and *eb*: for thus the Area of the one will be $\frac{1}{4}iaa$, and of the other $\frac{1}{4}ibb$.

CONSECTARY II.

CYlinders whoſe Altitudes are equal to the Diameters of their Baſes, are in proportion to one another as the Cubes of their Diameters; for the Cylinders will be $\frac{1}{4}ea^3$ and $\frac{1}{4}eb^3$, and the Cubes of the Baſes a^3 and b^3.

CONSECTARY III.

HENCE alſo (whatever be the Ratio of a Sphere to a * Cylinder of the ſame Diameter and Height; which we will hereafter demonſtrate, and which in the mean while we will denote by y) I ſay, hence Spheres which have the ſame Proportion to one another as theſe Cylinders, (*viz.* as $\frac{1}{4}ea^3$ to $\frac{1}{4}eb^3$, ſo $\frac{1}{4}yea^3$ to $\frac{1}{4}yeb^3$) will alſo (by *ConſeƐt.* 1.) be in the ſame proportion as the Cubes a^3 to b^3 of their Diameters; as is alſo evident from the Terms themſelves. p. 88.

* *Prop.* 18. *lib.* xii.

PROPOSITION XXXIII.

AN Angle † *at the Center of any Circle* ACB (*Fig.* 60.) *to an Angle at the Circumference which has the ſame Arch* ADB *for its Baſe, is as* 2 *to* 1. 80

† *Eucl. Prop.* 20. *lib.* iii.

Demonſtration.

The Truth of this has already appear'd from *Schol. Definit.* 10. N° 3. but here we will demonſtrate it otherwiſe in 13. its three Caſes, after *Euclid*'s way. In the firſt Caſe DE being conceived Parallel to CB, by *Def.* 11. *ConſeƐt.* 1 & 2. 14. 15. the external Angle ACB is= to the internal Angle ADE, and the Angle BDE, is equal to the alternate Angle DBC, *i.e.* to the other at the Baſe CDB, by *ConſeƐt.* 2. *Definit.* 19. 13. Therefore BDE is as 1, and CDE, *i.e.* ACB as 2.

In the ſecond Caſe, the whole ECB is double of the whole EDB, and the Angle ECA to be ſubſtracted, is double of the Angle EDA, to be ſubſtracted by Caſe 1. Therefore the Remainder ACB is alſo double of the Remainder ADB, by *Prop.* 26. In the third Caſe, the Part 83. ECA is double of the Part EDA, and alſo the Part ECB is double of the Part EDB, by Caſe 1. Therefore the whole ACB is double the whole ADB. Q.E.D.

CONSECTARIES.

I. HENCE all Angles ADB (*a*) in the ſame Segment are equal, and the Angle ADB (*Fig.* 61.) in a Semicircle is a right 100 one;

(*a*) *Eucl. Prop.* 21, 27, 31. *lib.* iii.

one; becauſe the Aperture ACB at the Center anſwering to it contains two right Angles: The Angle EDF, in a leſs Segment than a Semicircle, is greater than a right one; becauſe the Aperture at the Center EGHFC anſwering to it comprehends more than two right Angles. Laſtly, an Angle CDH, in a Segment greater than a Semicircle, is leſs than a right one; becauſe its double at the Center GCH is leſs than two right ones. All which we have already otherwiſe
p. 14. demonſtrated in *Schol. Def.* 10. N° 6.

* *Prop.* 32. *l.* I.

II. All the three Angles * of any Triangle ABD taken together, are equal to two right ones; becauſe they are the halves of the three at the Center
9. C, which always make four right ones, by *Definit.* 8. *Conſect.* 2.

III. Therefore any external Angle IAD, is equal to the two internal oppoſite ones at B and D; becauſe that, as well as they with the other BAD contiguous to them, make
ib. two right ones, by *Conſect.* 1. of the ſame *Definit.*

† *Prop.* 19. *lib.* I.

IV. And the greater Side of a Triangle, becauſe it ſtands on a † greater Arch of a circumſcribed Circle, does alſo neceſſarily ſubtend a greater Angle,
89. by virtue of *Conſect.* 1. of this.

PROPOSITION XXXIV.

p. 100. IN *equiangular Triangles* (ACB *and* abc, *Fig.* 62,) *the Sides about the equal Angles are proportional,* viz. *as* AB *to* BC, *ſo is* ab *to* bc, *and as* BC *to* CA *ſo is* bc *to* ca, *&c.* (*a*)

(*a*) *Prop.* 4. *lib.* vi.

Demonſtration.

For having deſcribed Circles thro' the Vertexes of each
10. Triangle, according to *Conſect.* 6. *Definit.* 8. becauſe the Angles A and *a*, B and *b*, C and *c*, are ſuppoſed equal, the Arches AB and *ab*, &c. will neceſſarily contain an equal Number of Degrees and Minutes, by the foregoing 33d *Prop.* and ſo alſo the Chords AB and *ab*, BC and *bc*, &c. will be each the ſame number of Parts of their Radii ZA and
12 *za*, by *Conſect.* 2. *Definit.* 10. that is, AC has as many Parts of AZ = 10000000, as *ac* will have of *az* = 10000000, *&c.* Therefore AC is to CB as *ac* to *cb*, &c., Q. E. D.

CONSECTARIES.

I. WHerefore by the ſame Neceſſity the Baſes AB and *ab* of ſuch Triangles, will be proportional to their Altitudes CD and *cd*, as being right Sines of like Arches

CB and cb, or rather CE and ce; and ſo for ſimilar or like Triangles (and conſequently alſo Parallelograms) we may rightly ſuppoſe that their Baſes are as a to ea, and their Heights as b to eb; tho' we muſt not immediately conclude on the contrary, that becauſe their Baſes and Altitudes are proportional, therefore they are ſimilar.

II. As alſo in ſimilar Parallelepipedons it will be manifeſt to any attentive Perſon, that the Baſes are in a duplicate Proportion of the Altitudes. For ſince the Planes of ſimilar Solids are equal in number, and ſimilar each to the other, if for 𝔄𝔅 (*Fig.* 63.) we put a, and for 𝔅ℭ, b; AB will = p.100.
ea and BC $= eb$; and ſo that Baſis will be to this as ab to $eeab$. Moreover, having let fall the Perpendiculars EH and 𝔈ℌ, the Triangles EBH and 𝔈𝔅ℌ are ſimilar, and by putting c for 𝔅ℭ, BE will be ec; putting alſo d for 𝔈ℌ, EH will conſequently be ed. But the Ratio of the Baſe ab to the Baſe $eeab$, is Duplicate of the Ratio of d to ed, by *Def.* 34. Wherefore in ſimilar Parallelepipedons, 51.
we may rightly ſuppoſe, that their Baſes are as ab to $eeab$, or as a to eea, and their Altitudes as d to ed.

SCHOLIUM I.

FROM this Propoſition flows, firſt of all, the chiefeſt Part of Trigonometry for the Reſolution of right-angled Triangles: For in any right-angled Triangle, ſince if one ſide, *e.g.* AB (*Fig.* 64.) be put for the whole Sine, or the Ra- 100
dius, the other BC will be the Tangent of the oppoſite Angle at A (and in like manner if CB be the whole Sine, BA will be the Tangent of the Angle C;) but if the Hypothenuſe AC be made Radius or the whole Sine, then the Side BC will be the right Sine of the Angle A, or the Arch CD deſcribed from the Center A, and AB the right Sine of the Angle C, or the Arch AE, deſcribed from the Center C, (we will omit mentioning the Secants, becauſe the buſineſs may be done without them) which all follow from *Def.* 10. Wherefore you may find, 11.

I. The Angles.

1. From the Sides.	by inferring	As one Leg to the other, ſo is the whole Sine, or the Radius, to the Tangent of the Angle oppoſite to the other Leg.
2. From the Hypoth. and one Side,		As the Hyp. to the Radius, ſo is the given Leg to the S. of the opp. Angl.

II. The

II. The Sides.

1. From the Hypoth. and Angles.	As the Radius to the Hypoth. ſo the Sine of the Angle, oppoſite to the Leg ſought, to the Leg itſelf.
2. From one Leg and the Angles.	As the Radius to the given Leg, ſo the Tangent of the Angle adjacent to it, to the Leg ſought.
3. From the Hypoth. and one of the Sides.	Having firſt found the Angles, it's done by Numb. 1. or by the *Pythagoric* Theorem.

III. The Hypothenuſe.

1. From the Angles, and one of the Legs.	As the Sine of the Angle, oppoſite to the given Leg, to that Leg, ſo is the Radius to the Hypoth.
2. From the Legs given.	Having firſt found the Angles, it is done by Numb. 1. or by the *Pythagorick* Theor.

III. Inverſly alſo, if two Triangles ABC and 𝔄𝔅ℭ (in the Figure of the preſent Propoſition) have one Angle of one equal to one Angle of the other (*e.g.* B and 𝔅) and the Sides that contain theſe equal Angles proportional (*viz.* AB to BC as 𝔄𝔅 to 𝔅ℭ) then the other Angles (A and 𝔄, C and ℭ) will be alſo equal, and the Triangles ſimilar; * for to the like Chords AB and 𝔄𝔅, BC and 𝔅ℭ, there anſwer, by the Hypoth. like or ſimilar Arches, *i. e.* equal as to the Number of Degrees and Minutes; and to theſe alſo there anſwer equal Angles both at the Periphery and Center.

* Eucl. prop. 6. lib. vi.

p. 100. IV. (*Fig.* 65. N°. 1.) If the * Sides of the Triangle BAC are cut by a Line DE, parallel to the Baſe BC, the Segments of thoſe Sides will be proportional, *viz.* AE will be to EC as AD to BD; for by reaſon of the Paralleliſm of the Lines DE and BC, the Triangles ADE and ABC are equi-angular: Therefore as the Whole BA to the Whole AC, ſo is the Part AD to the Part AE; and conſequently alſo the Remainder DB to the Remainder EC, as the Part AD to the Part EA, by *Prop.* 26. and alternatively by *Prop.* 23. as BD to AD, ſo is EC to EA. 83, 82.

* Eucl. 2. lib. 6.

SCHOLIUM II.

THere are ſeveral uſeful Geometrical Praxes that depend on this *Conſectary* and its *Propoſition:* 1. We are taught * by it to cut off any Part required (*e.g.* $\frac{1}{3}$) from a given Line AB, and ſo generally to cut or divide any given Line AC, in the ſame Proportion as any other given Line is ſuppoſed to be divided in D, (and conſequently into as many equal Parts as you pleaſe;) *viz.* in the firſt Caſe, if having any how drawn the Line AF, you take AD as 1, and make DB, 2, and having joined CB, draw the Parallel ED: for as AD to DB, ſo is AE to EC; that is, as 1 to 2, by this 4th *Conſect.* therefore AE is one third of the Whole AC, *&c.* p.92.

* *Eucl.* 9 & 10. *l.* 6.

II. We have a Rule (*a*) to find a third Proportional to two given right Lines AB and BC, (N°. 2. *Fig.* 65.) (or a fourth to three given ones) *viz.* draw AF at pleaſure, and make AD equal to BC, and join DB, and draw the Parallel EC: Then as AB to BC, ſo is AD (*i.e.* BC) to DE. Now if AD be not equal to BC, but to another, *viz.* a third Proportional, then by the ſame Reaſon DE will be a fourth Proportional. 100.

(*a*) *Eucl.* 11 & 12. *l.* 6.

III. Another Rule * to find a mean Proportional (Numb. 3.) between two given right Lines AC and CB; which is done by joining both the Lines together, and from the middle of the Whole AB deſcribing a Semicircle, and from C erecting the Perpendicular CD: For ſince the Angle ADB is a right one, by *Conſect.* 1. of the preceding *Propoſition*, and the two Angles at C are right ones, and thoſe at A and B common to the whole Triangle ADB, and to the two partial ones ACD and BCD, theſe two will be equi-angular and ſimilar to the great one, and conſequently to one another: Therefore by the preſent *Propoſition*, as AC to CD, ſo CD to CB; *Q.E.D.* And alſo as AB to BD, ſo BD to BC; and as AB to AD, ſo AD to AC, *&c.* ib. 89. 90.

* *Eucl.* 13. *lib.* 6. & 8. *lib.* 6.

IV. The Analytical Praxis of multiplying and dividing Lines by Lines, ſo that the Product or Quotient may be a Line; and alſo the Way of extracting the Roots of Lines: which *Des Cartes* gives us, *p.* 2. of his *Geom.* Aſſume ſome Line for Unity, *e. g.* AB (in *Fig.* 65. N°. 2.) then if AC is to be multiplied by AD, having joined BD, and drawn the Parallel CE, the Product will be AE; for it will be as 1, that is, AB, to the Multiplier AD, ſo is the Multiplicand 100.

tiplicand AC to the Product AE; or if AE is to be divided by AC, having joined EC, and drawn the Parallel BD, the Quotient will be AD; (for AC the Diviſor, will be to AE the Dividend, as Unity AB to the Quotient AD:) all which are evident from the Nature of Multiplication and Diviſion, and the precedent Praxes. As alſo taking CB (in p. 100. the ſame *Fig.* N°. 3.) for Unity, if the Square Root is to be extracted out of any other Line AC, this being joined to the Line CB = 1, ſo as to make one Line AB; and having deſcribed thereon a Semi-Circle, the Perpendicular CD will be the Root ſought, as being a mean Proportional be- p. 66. tween the two Extremes CB and AC, according to *Prop.* 17.

V. A right Line AG which divides * any given Angle A into two equal Parts (*Fig.* 66.) being prolonged, divides the Baſe BC in the ſame Proportion as the Legs of the Angle AB and AC have. For having prolonged CA to E, ſo that AE be = to AB; the Angles ABE and AEB 19. will be equal, by *Conſect.* 2. *Def.* 13. and conſequently alſo equal to each of the halves of the external Angle CAB, by 90. *Conſect.* 3. of the foregoing Propoſition. Therefore the 14. Lines AG and EB will be parallel, by *Conſ.* 1. *Def.* 11. Therefore as AC to AE, *i.e.* to AB, ſo is GC to GB, by 92. *Conſect.* 3. of this *Propoſition.* Q.E.D.

VI. Hence alſo it further follows, by converſion of the laſt Inference, that as AC + AB is to AC, ſo GC + GB (*i.e.* BC) to GC; and inverſly GC to BC as AC to AC + AB; and laſtly alternatively, GC to AC as BC to AC + AB.

N. B. *This laſt Inference follows alſo immediately from the preceding* Conſect. *For by reaſon of the Similitude of the Triangles* ACG *and* ECB, *as* GC *to* AC, *ſo is* BC *to* CE, i. e. *to* AC + AB.

SCHOLIUM III.

FRrom the two laſt Conſectaries there ariſe theſe two or three Practical Rules, the firſt whereof ſhews, how *by having the two Legs* AB *and* AC *given, and alſo the Baſe* BC, *to find the Segments* GC *and* GB, *made by the Biſection of the Intercrural Angle, viz.* by this Inference, according to *Conſect.* 6. As the Sum of the Sides to one Side, (*e.g.* AC) ſo is the Sum of the Segments of the Baſe, *i.e.* the whole Baſe, to one of the Segments, *viz.* that next the ſaid Side GC. 2. It ſhews on the contrary, how, *having the Baſe and one of its Segments given, and moreover the*

Sum

* Eucl. 3. lib. vi.

Sum of the Sides, to find ſeparately the Side AC *next the known Segment*; by inferring as the Sum of the Segments, or the Baſe BC to the Sum of the Sides, ſo is the given Segment GC to the ſought Side AC; or alſo, 3. *Having only the Baſe and Sum of the Sides given,* but not the Segment GC, *yet to expreſs its Proportion to the next Side* AC, by putting, (by *Conſect.* 6.) for GC the Value of the Baſe BC, and for AC the Value of the Sum AB + AC in the Quantities of the given Terms; the great Uſe of which laſt Rule will appear hereafter in the Cyclometry (or Quadrature of the Circle) of *Archimedes*.

VII. In any Triangle ABC, (*Vid. Fig.* of the preſent *Propoſition*) the Sides are to one another as the Sines of their oppoſite Angles: For they are as the Chords of the double Angles at the Center, by *Prop.* 33. therefore they p. 89. are alſo one to another as the halves of thoſe Chords, *i. e.* by *Definit.* 10. as the Sines of the half Angles. 11.

SCHOLIUM IV.

HEnce flow two new Rules of *Plane Trigonometry*, for *oblique-angled Triangles, viz.* to find,

I. The other Angles.

From two given Sides, and an Angle oppoſite to one of them. } by inferring { As the Side oppoſite to the given Angle is to the other Side, ſo is the Sine of the given Angle to the Sine of the Angle oppoſite to the other Side; which being given, the third is eaſily found.

II. The other Sides.

From one Side and the Angles given. } As the Sine of the Angle oppoſite to the given Side, to that Side; ſo is the Sine of the Angle oppoſite to the Side ſought, to the Side ſought.

So that this way we have reduced all the Caſes excepting one of Plane Trigonometry, and conſequently all *Euthymetry* to their original Foundations; (for in that caſe, where two Sides, and the included Angle are given, we may find the reſt by the Reſolution of an oblique-angled Triangle into two right-angled ones; and ſo it's done by the Rules we have deduced in *Schol.* 1.) I ſay, excepting one, in which from the three Sides of an oblique-angled Triangle given, you are required to find the Angles; the Rule to reſolve which we will hereafter deduce in the 2d *Conſect.* of

Prop. 45. from that Theorem which *Euclid* gives us, *lib.* 2. *Prop.* 13.

p. 100. VIII. Becauſe in the right-angled △ BAC (*Fig.* 67.) BC is to CA as CA to CD, by N°3. of the 2d *Schol.* of this *Prop.* the □CA will be = □CE, by *Prop.* 17. In like manner becauſe as CB to BA, ſo is BA to BD; the □BA will be = to □BE: Wherefore the two Rectangles BE and CE taken together, that is, the □ of the Hypothenuſe BC, will be = to the two Squares BA and CA taken together: Which is the very Theorem of *Pythagoras* demonſtrated two other ways in the *Schol.* of *Definit.* 13.

SCHOLIUM V.

AS this Theorem of *Pythagoras* furniſhes us with Rules for adding Squares into one Sum, or ſubſtracting one Square from another; ſo likewiſe it helps us to ſome Foundations whereon, among the reſt, the Structure of the Tables of Sines relies, *&c.* Whoſe Uſe we have already partly ſhewn in *Schol.* 1 & 4. *viz.* 1. If ſeveral Squares are to be collected into one Sum, having joined the Sides of two of them ſo as to form a right Angle, *e.g.* AB and BC (*Fig.* 68. N°1.) the Hypothenuſe AC being drawn, is the Side of a Square equal to them both; and if this Hypothenuſe AC be removed from B to D, and the Side of the third Square from B to E, the new Hypothenuſe DE will be the Side of a Square equal to the three former taken together. 2. If the Square of the Side MN (N°2.) is to be ſubſtracted from the Square of the Side LM; having deſcribed a Semicircle upon LM, and placed the other MN within that Semicircle, then draw the Line, LN and that will be the Side of the remaining Square. 3. Having the right Sine (*Fig.* 69.) EG of any Arch ED given (but how to find the primary Sines, we will ſhew in another place) you may obtain the Sine Complement CG or EF, by the preceding *Numb. viz.* by ſubſtracting the □ of the given Sine from the □ of the Radius; and moreover the verſed Sine GD, by ſubſtracting the Sine Complement CG from the Radius CD. 4. The Squares of the verſed Sine GD, and of the right Sine EG being added together, give the □ of the Chord ED of the ſame Arch, (which all are evident from the *Pythagorick Theorem*) and EH half of that, gives the right Sine of half that Arch. 5. From the right Sine EG you have the Tangent of that Arch, if you make as the Sine Complement CG to the right Sine GE, ſo is the whole Sine CD to the Tangent DI. 6. Laſtly, From theſe *Data* you may alſo have the Secants (if

re-

6. Laſtly, From theſe *Data* you may alſo have the Secants (if required) thus: As the Sine Complement CG to the Radius CE, ſo the Radius CD to the Secant CI; or as the right Sine EG to the Radius EC, ſo is the Tangent ID to the Secant IC; both which are evident by our 54th Propoſition. p. 90.

Conſect. IX. If the Quadrant of a Circle (CBEG, *Fig.* 70.) be inclined to another Quadrant (CADG) and two other perpendicular Quadrants cut both of them, *viz.* FBAG and FEDG, (the latter of which cutting them in the Extremities E,D) then if Perpendiculars be let fall from the common Sections E and B thro' them, the Planes of the perpendicular Quadrants, and the inclined Quadrant, (*viz.* EG and BH, as right Sines of the Segments EC and BC, and EI and BK, as right Sines of the Segments ED and BA) you will have two Triangles EIG and BKH right-angled at I and K, equiangular at G and H, (by reaſon of the ſame Inclination of the Plane CBEGC) and conſequently ſimilar, by our 54th Propoſition; wherefore as the Sine EG to the Sine EI, ſo the Sine BH to the Sine BK, or as EG to BH, ſo is EI to BK, and contrariwiſe. 100.

SCHOLIUM VI.

HENCE you have ſeveral Rules of *Spherical Trigonometry* for reſolving right-angled Triangles*. 1. In the right-angled △ ABC, having given the Hypothenuſe BC and the adjacent Angle ACB, to find the Leg AB oppoſite to this Angle, ſay, *As the whole Sine or the Radius* EG *to* BH *the Sine of the Hypoth. ſo is* EI *the Sine of the given Angle to* BK, *the Sine of the Leg ſought.* 2. The Hypothenuſe BC, and the Leg AB being given, to find the oppoſite Angle ACB, ſay, *As* BH *the Sign of the Hypoth. to the Radius* EC, *ſo is* BK *the Sine of the given Leg to* EI *the Sine of the Angle ſought.* 3. The Side AB and the Angle ACB, oppoſite to it, being given, to find the Hypothenuſe BC (ſuppoſing you know whether it be greater or leſs than a Quadrant) ſay, *As the Sine* EI *of the given Angle to the Radius* EG, *ſo is* BK *the Sine of the given Leg to* BH *the Sine of the Hypoth.* 4. In the right-angled △ EBF (which we take inſtead of ABC, that ſo we may not be obliged to change the Figure) one Leg EB and the Hypothenuſe BF being given, to find the other Leg EF, you may find its

* *Lansberg. Geom. Triang. lib.* 4. *Prop.* 12.

Complement, by ſaying, *As* BH *the Sine Complement of the given Side to the Radius* EG, *ſo is* BK *the Sine Complement of the Hypothenuſe, to* EI *the Sine Complement of Side ſought.* 5. Having both the Legs EB and EF given, the Hypothenuſe BF may be found; for its Compl. BA may be had by ſaying, *As the Radius* EG *is to* BH *the Sine Compl. of one Side* EB, *ſo is* EI *the Sine Compl. of the other Side* EF *to* BK *the Sine Complement of the Hypothenuſe.*

6. In the ſame right-angled Triangle EBF, one Leg EF, and the Angle EFB adjacent to it being given, to find the other oblique Angle EBF; *Firſt*, Produce BA to *f*, ſo that BF be a Quadrant, and then Af=BF Hypoth. and BC to *e*, ſo that BE be a Quadrant; and then Ce=EB, and AC to *d*, ſo that AD be a Quadrant, then will Cd=DA the Meaſure of the given Angle EFB. *Secondly*, From *d* thro' *e* and *f*, the Extremities of the Quadrants Bf and Be, let fall a Quadrant, that ſo the △ Cde may be right-angled, in which there are given the Hypoth. Cd= to the given Angle, and the Angle C= to the Compl. of the given Leg, (*viz.* to the Arch ED.) And ſo, *Thirdly*, the Side *de* muſt be ſought, as the Complement of the Arch *ef*, or of the Angle ſought ABC, or EBF; *viz.* by ſaying, according to the firſt Caſe of this, as the Radius to the Sine of the Hypoth. Cd (*i.e.* of the given Angle EFB) ſo is the Sine of the Angle dCe (*i.e.* DE the Compl. of the given Leg EF) to the Sine *de* (as the Compl. of the Angle fBe or EBF.)

7. In the ſame Triangle, having given the Side EF and the oppoſite Angle EBF (*i. e.* the Arch *ef*) to find the other Angle EFB; (that is, the Hypoth. Cd in the △ Cde) ſay by the third of this:

As the Sine of the Angle dCe (*i.e.* the Sine Compl. of the given Leg DE) is to the Radius, ſo the Sine of the Leg *de* (*i.e.* the Sine Compl. of the Angle EBF) to the Sine of the Hypoth. Cd (*i.e.* the Sine of the Arch DA or Angle EFB.)

8. Having the oblique Angles given to find either of the Legs, *viz.* EF; which may be done thus, by the ſecond of this:

As the Sine of the Hypoth. Cd (*i. e.* the Sine of the Angle at F) to the Radius, ſo is the Sine of *de* (*i. e.* the Sine Compl. of the Angle at B) to the Sine of the Angle dCe (*i. e.* the Sine Compl. of the Leg ſought EF.)

Conſect. X. The ſame being given as in *Conſect.* 7. if inſtead of the right Sines EI and BK, you erect the Perpendiculars DL and AM (*Fig.* 71.) then becauſe of the Similarity of the Triangles DGL and AHM, it will be as the Radius DG is to DL the Tangent of the Arch DE, ſo is AH the right Sine of the Arch AC, to AM the Tangent of the Arch AB; or as DG to AH, ſo is DL to AM, and contrariwiſe.

SCHOLIUM. VII.

HENCE flow the other Rules of *Spherical Trigonometry*, for reſolving right-angled Triangles, *viz.*

9. Having given the Leg AC in the △ ABC, and the adjacent Angle ACB, to find the other Leg AB, ſay, As the Radius DG to AH the Sine of the given Leg, ſo is the Tangent of the given Angle ACB to the Tangent AB of the Leg ſought.

10. Having given the Leg AB, and the oppoſite Angle at C, to find the other Leg AC (ſo you know whether it be greater or leſs than a Quadrant) ſay, As DL the Tangent of the given Angle to AB the Tangent of the given Leg, ſo is the Radius DG to the Sine of the Leg ſought, (*viz.* to AH.)

11. Both Legs being given, to find the Angles, ſay, As AH the Sine of one Leg to the Radius DG, ſo is AM the Tangent of the other Leg to the Tangent of the Angle at C, oppoſite to the ſame. 12. Moreover, in the right-angled Triangle EBF, the Hypothenuſe BF and the Angle EFB being given, to find the adjacent Leg EF, ſay, As AH the Sine Compl. of the given Angle to the Radius, ſo is the Tangent Compl. of the Hypoth. AM to DL the Tang. Compl. of the Leg ſought. 13. The Side EF and the adjacent Angle F being given, to find the Hypoth. BF, ſay, As the Radius to AH the Sine Compl. of the given Angle, ſo is DL the Tangent Complement of the given Leg to AM the Tangent Complement of the Hypoth. 14. The Hypoth. BF and one Side EF being given, to find the adjacent Angle F, ſay, As DL the Tangent Complement of the given Leg to the Radius, ſo is the tangent Complement of the Hypoth. AM to AH the Sine Compl. of the Angle ſought. 15. Having given the Hypoth. BF (*i. e.* the Arch A*f*, or the Angle at *d*) and either of the oblique Angles at F, to find the other Angle EBF, by help of the new Triangle C*de*, ſay, by the 12th of this:

As the Sine Complement of the Angle C*de* (*i. e.* the Sine Complement of the Hypoth. AH) to the Radius, ſo is the Tangent Compl. of the Hypoth. C*d* (*i. e.* the Tang. Compl of the given Angle) to the Tang. Compl. of the Side *de* (*i.e.* to the Tang. of the Angle ſought ABC or EBF.)

16. The Oblique Angles being given, to find the Hypoth. (BF, or the Arch A*f*, or the Angle C*de*) it is done by the 14 of this *Schol.*

For as the Tang. Compl. of the Leg *de* (*i.e.* the Tangent of the Angle ABC or EBF) to the Radius, ſo is the Tangent Complement of the Hypoth. C*d* (*i.e.* the Tangent Complement of the other Angle EFB) to the Sine Compl. of the Angle C*de* (*i.e.* the Sine Compl. of the Hypoth. BC ſought.)

Hence we have now, with *Lansbergius*, ſcientifically reſolved all the Caſes of Right-angled Triangles, but much more compendiouſly; the Reſolution of Oblique-angled ones only now remaining.

ConſeƐt. XI. In Oblique-angled Spherical Triangles, as well as Right-angled ones, the Sines of the Angles are directly proportional to the Sines of the oppoſite Sides. 1. This
p.97. is evident of Right-angled Triangles from N°3. *Schol.* 6. and
ib. from the 9th *ConſeƐt.* For as the Sine of the Angle A (*Fig.* 72.) to the Sine of BD, ſo is the Radius (*i. e.* the Sine of the Angle D) to the Sine of AB. In like manner, as the Sine of the Angle B to the Sine of AD, ſo is the Radius (that is, the Sine of the Angle D) to the Sine of AB. 2. The ſame is immediately evident of an Oblique-angled Triangle ABC, reſolved into two Right-angled ones. For,

The Sine of the Angle A is to the Sine of BD, as the Sine of the Angle D to the Sine of AB: and alſo

The Sine of the Angle C is to the Sine of BD, as the Sine of the Angle D to the Sine of BC, by N°1.

Now in each Proportion the Means are the Sines of BD and D; therefore the Rectangles of the Extremes, *viz.* of the Sines of AB into the Sine of A, and of the Sine of BC into the Sine of C, will be equal to one another, ſince the
p.68. Rectangles under the ſame Means are equal, by *Prop.* 18.
ib. therefore by *Prop.* 19. as the Sine of A to the Sine of BC, ſo is the Sine of C to the Sine of AB. Q. E. D.

SCHOLIUM VIII.

THAT the Sines of the Angles of Oblique Spherical Triangles are proportional to the Sines of their oppoſite Sides, it may be ſhown alſo thus: becauſe the Sine of the Angle A is to the Sine of BD, as the Sine of the Angle D to the Sine of AB, call the firſt *a*, the ſecond *ea*, the third

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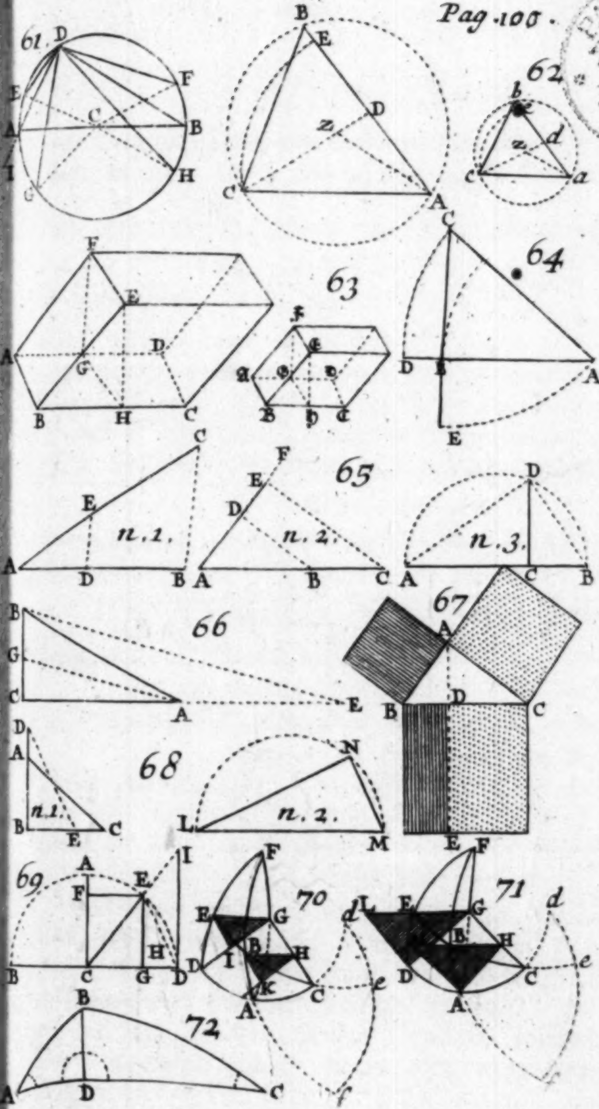

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third *b*, the fourth *eb*; and becauſe the Sine of the Angle C (which we call *c*) is likewiſe to the Sine of BD (*i.e.* to *ea*) as the Sine of D (*i.e.* *b*) to the Sine of BC, (which will conſequently be $\frac{eab}{c}$) it will be manifeſt, by multiplying the Means and Extremes, that the Sine of the Angle A is to the Sine of BC, as the Sine of the Angle C to the Sine of AB; *i.e.* as *a* to $\frac{eab}{c}$, i. e. as *c* to *eb*, ſince each of whoſe Rectangles is *eab*. Therefore as by the preſent and precedent *Conſect.* 7. it is univerſally true, *That in any Triangle, whether Right-lined or Spherical, Right-angled or Oblique-angled, the Sides, or their Sines, are to one another, as the Sines of their oppoſite Angles*; ſo alſo hence flow two new Rules of *Spherical Trigonometry* for Oblique-angled Triangles, like thoſe we found in *Schol.* 4. *viz.* to find,

I. *The other Angles.*

From having two Sides, and an Angle oppoſite to one of them, given;	by inferring	As the Sine of the Side oppoſite to the given Angle to the Sine of the other Side, ſo is the Sine of the given Angle to the Sine of the Angle ſought.

II. *The other Sides.*

From one Side and the Angles given,	by inferring	As the Sine of the Angle oppoſite to the given Side, to the Sine of that Side, ſo is the Sine of the Angle oppoſite to the Side ſought, to the Sine of the Side ſought.

And thus we have reduced all the Caſes and Rules of Spherical Trigonometry to their original Fountains, except two Caſes; *viz.* when from three Sides given, the Angles, or from three Angles the Sides are ſought: to reſolve which, we are ſupplied with Rules from the following *Conſectary*. For from two Sides given, and the interjacent Angle, or two Angles and their adjacent Side, we may find the reſt in Oblique-angled Triangles, by reſolving them into two Right-angled ones; and ſo by the Rules we have deduced in *Schol.* 6, & 7.

Conſect. XII. In the given Oblique-angled Spherical Triangle ABC (*Fig.* 73.) whoſe Sides are unequal, and each leſs than a Quadrant, having produced the Sides AB and AC till they become the Quadrants AD and AE, and beſides effected what the Figure directs; then will

The Arch DE be the Meaſure of the Angle A, and let AF = AC, and ſo FB is the difference of the Sides AB and AC.

BC = BG, and ſo GF is the difference between the third Side, and FB the difference of the other two.

But now, 1. As EH or DH to CM or FM, ſo will PH be to NM, (by reaſon of the Equi-angular Triangles EPH and CNM;) therefore by *Prop.* 26. ſo will alſo DP be to FN. Make therefore DH = *a*, FM = *ea*, DP = *b*, FN = *eb*.

AI the right Sine of the Side AB.

CM the Sine of the Side AC.

GL the Sine of GB, or of the Side BC.

FK the right Sine of the Arch FB.

BI the verſed Sine of AB.

BL the verſed Sine of GB or BC.

BK the verſed Sine of FB.

KL or NO the Difference of the verſed Sines laſt mentioned.

EP the right Sine, and DP the verſed Sine of the Arch DE.

CN the right Sine, and FN the verſed Sine of the Arch FC.

2. Becauſe the Triangles FNO and HAI are ſimilar, (ſince FNO is ſimilar to the △ FKQ, and FKQ to the △ HQM, by reaſon of the Vertical Angles at Q; and HQM alſo to the △ HAI, by reaſon of the common Angle at H) therefore

$$\left.\begin{matrix}AH\\ \text{or } DH\end{matrix}\right\} : AI :: FN : NO$$

$$a : oa :: eb : oeb$$

Whence, 3. It is evident that

$$\overline{DH}^{2} : FM \times AI :: DP : NO$$

$$aa : oeaa :: b : oeb$$

$$\text{that is, } a : oea :: b : oeb$$

$$\text{and inverſly } oea : \left.\begin{matrix}DH\\ a\end{matrix}\right\} :: \begin{matrix}NO : DP\\ oeb : b\end{matrix}$$

SCHOLIUM IX.

SINCE therefore the Radius DH or *a* is known, and alſo NO the Difference of the verſed Sines BL and BK, it is evident that DP, the verſed Sine of the Angle A, will be known alſo; ſuppoſing that the firſt Quantity *oea* be likewiſe known. But this may be had by another antecedent Inference, if you make

$$\begin{array}{c}\left.\begin{array}{r}DH\\ \text{or } AH\end{array}\right\} : FM :: AI : \\ a : ea :: oa : \end{array}\Bigg\}\ oea.$$

Hence therefore ariſes, (1.) The Rule: *By having the three Sides of an Oblique-angled Triangle given, to find any of the Angles*; *viz.* by inferring,

1. As the Radius to the right Sine of one of the comprehending Sides AC; ſo is the Sine of the other Side AB, to a fourth Sine.

2. As this fourth Sine to the Radius, ſo is the Difference of the verſed Sines of the third Side BC, and the Difference of the other Sides, to the verſed Side of the Angle ſought, *viz.*

$$oea : a :: \left\{\begin{array}{l} NO : DP. \\ oeb : b.\end{array}\right.$$

But ſince the Sides of a Spherical Triangle may be changed into Angles, and contrariwiſe, by continuing out the Sides; [as if the Side AB of the given Triangle ABC, (*Fig.* 74.) be continued to a whole Circle, and the reſt to Semicircles ACE, BCD, from the Poles *b* and *c*, and likewiſe the Semicircle HI from the Pole A, and the Semicircle FG from the Pole B, and the Semicircle EA from the Pole C, you'll have a new Triangle *abc*, the three Angles of which will be equal to the three Sides of the former ABC; as the Angle *a*, or its Meaſure IG, is equal to the Side AB, by reaſon each with the third Arch AG makes a Quadrant: but the Meaſure of the Angle *b*, is the Side AC (*viz.* in this Caſe wherein the Side AC is a Quadrant, in the other wherein it would be greater or leſs than a Quadrant, it would be the Meaſure of the Compl. of the Angle; for then the Semicircle H*ab* deſcribed from the Pole A, would not paſs thro' C, but beyond or on this ſide of C. See *Pitiſc. lib.*1. *prop.* 61. *p.m.* 25.) the Angle *c*, or its Meaſure KL, is equal to the Side BC, becauſe with the third KC they make the Quadrants BK and CL.] Therefore,

(2.) *The three Angles of the Oblique-angled Triangle abc, being given, you may find any Side*; e.g. *ac*, if there be ſought the Angle ABC, or rather its Complement KBF, or its Meaſure FK $= ac$, from the three Sides given of the △ ABC, by the preceding Rule, by inferring,

1. As the Radius to the Sine of one Side AB comprehending the Angle, (*i. e.* of one Angle *a* adjacent to the Side ſought) ſo is the Sine of the other Side BC (*i.e.* of the other Angle *c*) to a fourth.

2. As that fourth to the Radius, ſo is the Difference of the verſed Sines of the third Side AC, and the Difference of the other Sides (*i.e.* of the third Angle *b*, and the Difference of the others) to the verſed Sine of the comprehended Angle, or its Complement to a Semicircle, (*i. e.* of the Side ſought *ac*.)

PROPOSITION XXXV.

* Eucl. 19 & 20. lib. vi.

*Similar Plane Figures * are to one another in the Duplicate Proportion of their Homologous Sides.*

Demonſtration.

For, 1. The Baſes of two ſimilar Triangles or Parallelograms, (for which any two Homologous Sides, *e.g.* AB and 𝔄𝔅, *Fig.* 75. may be taken) and their Perpendiculars DE and 𝔇𝔈, by *Conſect.* 1. *Prop.* 34. will be as *a* to *ea*, and *b* to *eb*. Therefore the Parallelograms and Triangles themſelves will be as *ba* to *eeba*, by *Conſect.* 7, & 8. *Def.* 12. *i. e.* by *Def.* 34. in the duplicate Ratio of their Perpendiculars or aſſumed Sides: This is moſt conſpicuous in Squares, becauſe putting *a* for the Side of one, and *ea* for the other, they are to one another as *aa* to *eeaa*.

p. 109. 90.

2. Similar Polygons are reſolved into ſimilar Triangles, ſince the Triangles ABC and 𝔄𝔅ℭ, as alſo AED and 𝔄𝔈𝔇 are Equi-angular, by *Conſect.* 3. *Prop.* 34. but CAD and ℭ𝔄𝔇 are alſo Equi-angular, becauſe each of their Angles is the remainder of equal ones, after equal ones are taken from them. Wherefore the firſt Triangles are in the duplicate Proportion of the Sides BC and 𝔅ℭ; the ſecond, of the Sides CD and ℭ𝔇; the third, of the Sides DE and 𝔇𝔈, &c. *i.e.* (ſince by the Hypoth. BC has the ſame Ratio to 𝔅ℭ, as CD to ℭ𝔇, and DE to 𝔇𝔈) each to each is in the duplicate Proportion of the Sides BC to 𝔅ℭ, or CD to ℭ𝔇, by the firſt of this. Therefore by a Syllepſis, the whole Polygons are in the duplicate Proportion of the ſaid Sides: Which is the ſecond thing to be demonſtrated.

3. Circles and their ſimilar Sectors, are as the Squares of their Diameters, by *Prop.* 32. therefore they are in the du-

plicate Proportion of them, by the firſt of this; which is the third thing to be demonſtrated: Therefore ſimilar plane Figures, *&c.* Q. E. D.

CONSECTARIES.

I. THerefore two ſimilar plane Figures are one to another, as the firſt Homologous Side, to a third Proportional, by virtue of *Definit.* 34. p.251..

II. Any Figures deſcribed on four proportional Lines, ſo as each two of them be * ſimilar, are likewiſe proportional, and contrariwiſe; for if the ſimple Ratio's or Proportions of Lines be the ſame, their duplicate Proportions will be the ſame alſo, and reciprocally.

* *Eucl. Prop.* 22. *lib.* vi.

SCHOLIUM.

AS this ſecond *Conſectary* confirms *Prop.* 22. and its *Scholium*, ſo the firſt teaches us a twofold Geometrical Praxis. 1. A way to expreſs the Proportion of ſimilar Figures by two right Lines, *viz.* by finding a third Proportional to their Homologous Sides: For as the Side of the firſt Figure to this third Proportional, ſo will the firſt Figure be to the ſecond. 2. A way to augment or diminiſh any given Figure in a given Ratio, or Proportion, *viz.* by finding a mean Proportional between any Side of the given Figure, and another Line which ſhall be to that in a given Proportion, and then by deſcribing thereon a ſimilar or like Figure. 81.

PROPOSITION XXXVI.

SImilar or like ſolid Figures, are to one another in the triplicate Proportion of their Homologous Sides.

Demonſtration.

For, 1. The ſimilar Baſes of two ſimilar Parallelepipedons (and conſequently alſo of Priſms and Cylinders, by *Conſect.* 4 and 5. *Definit.* 16. and alſo of Pyramids and Cones, by *Conſect.* 3 and 4, of *Definit.* 17.) are, as *ab* to *eeab*, by † the preceding Propoſition, and their Altitudes, as *c* to *ec*, by *Conſect.* 2. *Prop.* 34. 25. 27. 91.

† *Eucl. Prop.* 12. *l.* xii. *of Cones and Cylinders.*

Therefore Parallelepipedons, Cylinders, and Priſms (and ſo the third Part of theſe, *viz.* Cones and Pyramids) will be as *abc* to e^3abc, by *Conſect.* 3, 4, 5. *Definit.* 16. *i. e.* by *Definit.* 34. and *Conſect.* 1, and 2. *Prop.* 34. 25. 51, 90 they will be in the triplicate Proportion of their Perpendiculars or Homologous Sides, which is the firſt thing to be demonſtrated. This is very conſpicuous in Cubes, ſince putting *a* for the Side of one, and *ea* for the Side of the other, they are to one another, as a^3 to e^3a^3.

2. Similar Polyhedrons may be reſolved into Pyramids of ſimilar Baſes and Altitudes, which is evident of regular ones, p.32. from the *Conſect.* of *Definit*. 21. and cannot be difficult to underſtand alſo of irregular ones; becauſe the like Inclination of their Planes every where ſimilar and equal in number, neceſſarily require that the whole Altitudes of ſimilar polyhedrous Solids, as well as ſimilar Parallelepipedons, by 91. *Conſect*. 2. *Prop*. 34. ſhould be in the ſubduplicate Proportion of their Baſes; and ſo theſe being divided in C and 108. ℭ (*Fig*. 76. N° 1.) the Parts of their Heights GC and 𝔊ℭ, will be in the ſame Proportion: Whence, *e. g.* two Pyramids ſtanding on ſimilar Baſes ABDEF and 𝔄𝔅𝔇𝔈𝔉, and having like Altitudes GC and 𝔊ℭ, will neceſſarily be like or ſimilar; and the ſame thing may be likewiſe judged of others.

Or yet, to ſhew it more evidently, the polyhedrous Solids may be reſolved into like triangular Priſms; for, *e. g* each of the Triangles of their ſimilar Baſes ABDEF and 𝔄𝔅𝔇𝔈𝔉 (N° 2.) are ſimilar, *viz*. 𝔞𝔟𝔣 and *abf*, 𝔄𝔅𝔉 and ABF, by the preceding *Prop*. N° 2. The Planes 𝔄𝔅𝔟𝔞 and AB*ba*, alſo 𝔄𝔞𝔣𝔉 and A*af*F, are ſimilar by the Hypoth. and conſequently alſo the Planes 𝔅𝔟𝔣𝔉 and B*bf*F (for 𝔣𝔟 is to 𝔟𝔞 as *fb* to *ba*, and alſo in the one 𝔟𝔞 to 𝔟𝔅, as *ba* to *b*B in the other; therefore by equality of Proportion, as 𝔣𝔟 to 𝔟𝔅, ſo is *fb* to *b*B, &c.) and ſo the whole triangular Priſms will be ſimilar, by *Definit*. 35. and ſo of others. Therefore ſimilar Polyhedrous Solids will be in the ſame Proportion as ſimilar Pyramids, or triangular Priſms, *i. e.* by the firſt of this in the triplicate Proportion of their Sides.

3. Spheres are as the Cubes of their Diameters *, by *Conſect*. 3. *Prop*. 32. Therefore, by the firſt of this, they are as a^3 to e^3a^3. Therefore ſimilar or like Solids are in the triplicate Proportion of their Homologous Sides. Q. E. D.

* *Eucl.* 18. *lib.* 12.

◆

CHAP.

CHAP. VI.

Of the Proportions of Magnitudes of divers kinds compared together.

PROPOSITION XXXVII.

THE *Parallelogram* ABCD (Fig. 77. N. 1.) *is to the Triangle* BCD *upon the ſame Baſe* DC, *and of the ſame Height, as* 2 *to* 1. This has been already demonſtrated in *ConſeƐt.* 3. *Definit.* 12. But here we ſhall give you another

p. 108.

16.

Demonſtration.

Suppoſe, 1. the whole Baſe CD divided into four equal Parts by the tranſverſe parallel Lines EG, HK, LN, then (by reaſon of the Similarity of the Triangles DGF, DKI, DNM, DCB) GF will be 1, KI, 2, NM, 3, CB, 4; and having again biſected the Parts of the Baſe, the Indiviſibles or the Portions of the Lines drawn tranſverſly thro' the Triangle will be 1, 2, 3, 4, 5, 6, 7, 8. And ſo again, *ad infinitum*, all-along in an arithmetical Progreſſion, beginning from the Point D, as 0; to which the like Number of Indiviſibles equal to the greateſt, always anſwer in the Parallelogram, *viz.* the Line BC. Wherefore by the 4th *ConſeƐt.* of *Prop.* 16. all the Indiviſibles of the Triangle, to all thoſe of the Parallelogram taken together, *i. e*, the Triangle itſelf to the Parallelogram, is as 1 to 2. Q. E. D.

SCHOLIUM.

NOW if any one ſhould doubt whether the Triangle or Parallelogram may be rightly ſaid to conſiſt of an infinite Number of indiviſible Lines; inſtead of Lines, he may, with Dr. *Wallis*, conceive infinitely little Parallelograms of the ſame infinitely little Height, and it will do as well. For having cut the Baſe (N° 2.) into four equal Parts by tranſverſe Parallels, there will be circumſcribed about the Triangle ſo many Parallelograms of equal height, being in the ſame Proportion as their Baſes, by *Prop.* 28. *i.e.* increaſing in arithmetical Progreſſion. In the next Biſection, there will ariſe eight ſuch Parallelograms approaching nearer to the Triangle, in the next 16, &c. So that at length infinite ſuch Parallelograms of infinitely leſs height, and ending in the Triangle itſelf, will conſtitute or make an infinite Series of arithmetical Proportionals, beginning not from 0, but 1; to which there will anſwer in the Parallelogram infinite little Parallelograms of the ſame Height, equal to the greateſt. *p. 78.* Whence it again follows, by *Conſect.* 9. *Prop.* 21. that the one Series is to the other, *i. e.* the Triangle to the Parallelogram as 1 to 2; which being here thus once explained, may be the more eaſily applied to Caſes of the like nature hereafter.

CONSECTARIES.

p. 124. I. IN like manner ſince in the Circle (*Fig.* 79.) the Peripheries at equal Intervals from one another, as ſo many Elements of the Circle, increaſe in arithmetical Progreſſion; the Sum of theſe Elements, *i.e.* the Circle itſelf, will be to the Sum of as many Terms equal to the greateſt Periphery; *i.e.* to a cylindrical Surface, whoſe Baſe is the greateſt Periphery, and its Altitude the Semidiameter, as 1 to 2.

II. Hence the curve Surface of a Cylinder circumſcribed about a Sphere, *i. e.* whoſe Altitude is equal to the Diameter, is the Quadruple of the Baſe.

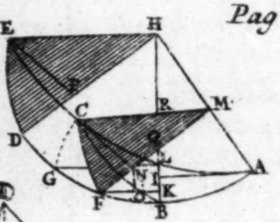
III. Alſo the Sector *bac* of the Circle to a cylindrical Surface, whoſe Baſe is the Arch *bc*, and its Altitude the Semidiameter *ac*, is as 1 to 2.

IV. And becauſe the Surface of the Cone BCD is to its circular Baſe, as BC to CA, *i. e.* as the $\sqrt{2}$ to 1, by *Schol.* *p. 67* *Prop.* 17, the cylindrical Surface, the conical Surface, and the circular one we have hitherto made uſe of, will be as 2, $\sqrt{2}$ and 1, and conſequently continually proportional.

SCHOLIUM.

ALL which may likewiſe abundantly appear this way; for by putting *a* for the Diameter of the Circle, $\frac{1}{2}a$ for the Semidiameter, and *ea* for the Circumference, you *p. 49.* will have the Area of the Circle equal to $\frac{1}{4}eaa$, by *Conſect* 1. *Definit.* 31. and multiplying the Height of the Cylinder AB, *i. e.* $\frac{1}{2}a$, by the Periphery *ea*, you will have the cylindrical *p. 29.* Surface $\frac{1}{2}eaa$, by *Conſect.* 6. *Definit.* 18. as now is evident alſo by *Conſect.* 1, 2 and 3. Now if you would alſo have the Surface of the Cone, ſince its Side, by the *Pythagorick Theorem*,

73



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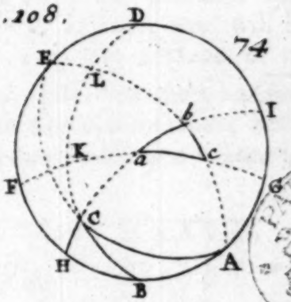

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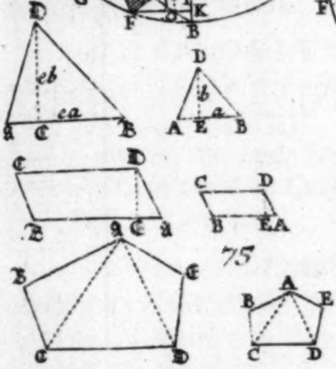
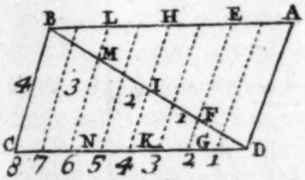
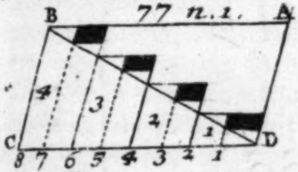




77 n.1.



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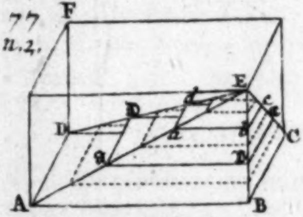

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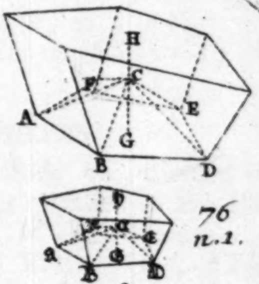

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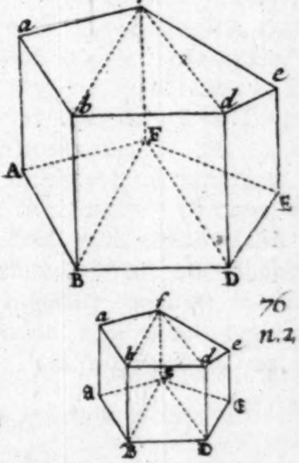

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Theorem, is $\sqrt{\frac{1}{2}aa}$ and the half of that $\frac{1}{2}\sqrt{\frac{1}{2}aa}$, *i. e.* (by N° 2. of *Schol. Prop.* 22.) $\sqrt{\frac{1}{8}aa}$; and this half being multiplied by the Periphery of the Baſe *ea*, you will have (by virtue of *Conſect.* 4. *Definit.* 18.) the Surface of the Cone $ea\sqrt{\frac{1}{8}aa}$, *i.e.* (by the *Schol.* juſt now cited) $\sqrt{\frac{1}{8}eea^4}$: So that now appears alſo the 4th *Conſect.* of this; becauſe the Rectangle of the Extremes $\frac{1}{2}eaa$ and $\frac{1}{4}eaa$ is $\frac{1}{8}eaa^4$, as well as the Square of the Mean.

PROPOSITION XXXVIII.

A *Parallelepipedon* * BF (Fig. 77. N° 3.) *is to a Pyramid* ABCDE, *upon the ſame Baſe* BD, *and of the ſame Height*, *as* 3 *to* 1.

* *Eucl. Prop.* 7. *Coroll. lib.* xii.

This was demonſtrated in *Conſect.* 3. *Definit.* 17. but here we ſhall give you another

Demonſtration.

Suppoſe, (1.) the whole Altitude BE, divided into three equal Parts, by tranſverſe Planes parallel to the Baſe, then (by reaſon of the Similitude of the Pyramids *abcd*E, 𝔄𝔅ℭ𝔇𝔈, and ABCDE) the Baſes *abcd*, 𝔄𝔅ℭ𝔇 and ABCD, by *Conſect.* 2. *Prop.* 34. and *Conſect.* 3. *Def.* 17. will be in the duplicate Proportion of the Altitudes, *i.e.* in the duplicate arithmetical Progreſſion 1, 4, 9. And again, (2.) biſecting the Parts of the Altitude, the quadrangular Sections now double in Number (as the Indiviſibles or Elements of the propoſed Pyramid) will be as 1, 4, 9, 16, 25, 36, &c. *ad infinitum*, all along in a duplicate arithmetical Proportion; while in the mean time there anſwer to them as many Elements in the Parallelepipedon, each equal to the greateſt ABCD, wherefore by *Conſect.* 10. *Prop.* 21. all the Indiviſibles of the Pyramid taken together, will be to all the Indiviſibles of the Parallelepipedon alſo taken together, *i. e.* the Pyramid itſelf to the Parallelepipedon, as 1 to 3. Q.E.D.

p.91. 27.

79.

CONSECTARY.

THIS Demonſtration may be eaſily accommodated to all other Pyramids and Priſms, and alſo to Cones and Cylinders †, ſince here alſo (*Fig.* 78.) the circular Planes *ba*, 𝔅𝔄, BA, are as the Squares of the Diameters, and ſo as 1, 4, 9. and ſo likewiſe all the other Elements of the Cone, by continual Biſection, are in a duplicate arithmetical Progreſſion; when in the mean

124.

† *Eucl. Prop.* 10. *lib.* xii.

mean time there anſwer to them in the Cylinder as many Elements equal to the greateſt BA, *&c.*

PROPOSITION XXXIX.

A *Cylinder is to a Sphere inſcribed in it,* i. e. *of the ſame Baſe and Altitude, as* 3 *to* 2.

Demonſtration.

p. 124. 1. Suppoſe (*Fig.* 80.) GH the half Altitude (for the ſame Proportion which will hold when demonſtrated of the half Cylinder AK, and Hemiſphere AGB, will alſo hold of the whole Cylinder and the whole Sphere) to be divided into three equal Parts, then will AH, C 1, E 2, be mean Proportionals between the Segments of the Diameter, by *Prop.*
93. 34. *Schol.* 2. N° 3. and ſo by *Prop.* 17. the Rectangles
66. LHG, L 1 G, L 2 G, equal to the Squares AH, C 1, E 2, will be in order, as 9, 8, and 5. In like manner, 2. Having biſected the former Parts of the half, the ſix Squares cutting the Sphere croſs-ways, will be found to be as 36, 35, 32, 27, 20, 11, *&c.* in the Progreſſion we have ſhewn at
80. large in *Conſect.* 12. *Prop.* 21. Wherefore ſince all the Indiviſibles of the Hemiſphere, *viz.* the circular Planes anſwering to the Squares of the ſaid tranſverſe Diameters, have the
88. ſame Proportion of Progreſſion, by *Prop.* 32. and there anſwer to them the like Number of Elements in the Cylinder equal to the greateſt AH: All theſe taken together, will be to all the others taken together, *i. e.* the whole Cylinder AK to the whole Hemiſphere AGB, by
80. virtue of the aforeſaid *Conſect.* 12. as 3 to 2*.
Q. E. D.

* *Archim.* 32. (*al.* 31.) *lib.* i. *de Sphær.*

SCHOLIUM I.

H*Onoratus Faber* elegantly deduces this *Prop. à priori,* from the Generation of the Solids, in his *Synopſis Geom. p.* 318. (which alſo *Carolus Renaldinus* performs, from the ſame common Foundation in *lib.* 1. *de Compoſ.* and *Reſol. p.* 301. but after a more obſcure way by a Demonſtration further fetch'd)
124. after this Method: The whole Figure (81) AL being turned round about BL, the Quadrant ADLB will deſcribe an Hemiſphere, the Square AL a Cylinder, and the Triangle BML a Cone, all of the ſame Baſe and Altitude. Since therefore Circles are as the Squares of their Diameters, by
88. *Prop.* 32. and the Square of GE = to the Squares of GD and GF taken together (for the Square of GF, *i. e.* □ of GB,

GB + □ of GD is = □ of BD, or □ of BA, or □ of GE, by the *Pythag. Theor.*) and ſo the Circle deſcribed by GE will be = to two Circles deſcribed by GD and GF taken together; then taking away the common Circle deſcribed by GD, there will remain the Circle deſcribed by GF within the Cone, equal to the *Annulus* or Ring deſcribed by DE about the Sphere. And ſince this may be demonſtrated after the ſame way in any other caſe, *viz.* that a Circle deſcribed by *gf*, will be equal to an *Annulus* deſcribed by *de*; it will follow, that all the Rings or *Annuli* deſcribed by the Lines DE or *de* (*i.e.* all that Solid that is conceived to be deſcribed by the trilinear Figure ADLM turned round) will be equal to all the Circles deſcribed by GF or *gf* (*i.e.* to the Cone generated by the Triangle BLM;) and ſo as the Cone is $\frac{1}{3}$ Part of the Cylinder generated by AL, according to *Conſect.* of *Prop.* 38. ſo alſo the Solid made by the p. 109.
Trilinear Solid ADLM (*viz.* the Exceſs of the Cylinder above the Sphere) will be $\frac{1}{3}$ of the Cylinder, and conſequently the Hemiſphere is $\frac{2}{3}$ of it. Q. E. D.

CONSECTARIES.

I. HENCE you have a further Confirmation of *Conſect.* 2. 89.
Prop. 32. and *Prop.* 36. N. 3. 106.

II. Hence alſo naturally flows a Confirmation of *Conſect.* 2. 30.
Definit. 20. and conſequently the Dimenſion of the Sphere, both as to its Solidity and Surface. For putting *a* for the Diameter of the Sphere and circumſcribed Cylinder, and *ea* for the Circumference, the Baſe of the greateſt Circle will be $\frac{1}{4}eaa$, and that multiplied by the Altitude, gives $\frac{1}{4}ea^3$ for the Cylinder. Therefore by the preſent Propoſition, $\frac{1}{6}ea^3$ gives the Solidity of the Sphere (by making as 3 to 2, ſo $\frac{1}{4}$ to $\frac{1}{6}$.) This divided by $\frac{1}{6}a$, by vertue of *Conſect.* 1. of the afore- 31.
ſaid *Def.* 20. and *Conſect.* 3. *Definit.* 17. will give the Sur- 27.
face of the Sphere *eaa*.

III. Therefore the * Surface of the Sphere *eaa*, is manifeſtly Quadruple of the greateſt Circle $\frac{1}{4}eaa$.

* *Archim. lib.* 1. *de Sph. & Cylind. Prop.* 31. (*al.* 30.)

IV. The Surface of the Cylinder, without the Baſes, found by multiplying the Altitude *a* by the circular Periphery of the Baſe *ea*, will be *eea*, equal to the Surface of the Sphere.

V. Adding therefore the two Baſes, each whereof is $\frac{1}{4}eaa$, the whole Surface of the Cylinder $1\frac{1}{2}eaa$, will be to the Surface of the Sphere *eaa* as 3 to 2.

VI. The Square of the Diameter *aa* to the Area of the Circle $\frac{1}{4}eaa$, is as *a* to $\frac{1}{4}ea$, *i.e.* as the Diameter to the 4th Part of the Circumference.

VII. A Cone of the ſame Baſe and Altitude with the
p. 111. Sphere and Cylinder, by *Conſect.* 2. of this and the *Conſect.*
109. of *Prop.* 38. will be $\frac{1}{12}ea^3$, and of the Cylinder $\frac{1}{4}$ or $\frac{3}{12}ea^3$. Therefore a Cone, Sphere, and Cylinder, of the ſame Height and Diameter, are as 1, 2, 3. The Cone therefore is equal to the Exceſs of the Cylinder above the Sphere; as
110. is otherwiſe evident in *Scholium* 1. of this.

SCHOLIUM II.

AND thus we have briefly and directly demonſtrated the chief Propoſitions of *Archimedes*, in his firſt Book *de Sphær. & Cylind.* which he has deduced by a tedious *Apparatus*, and only indirectly. And now if you have a mind to ſurvey the longer and more perplex'd way of *Archimedes*, and compare it with this ſhorter Cut we have given you; take it thus: *Archimedes* thought it neceſſary firſt of all to premiſe this *Lemma*, viz. *That all the conical Surfaces of a conical Body made by the Circumvolution of the Polygon,*
.124. *or many-angled Figure* ABCDE, *&c.* (Fig. 82.) *inſcribed*
p. 29. *in a Circle, according to* Definit. 19. *taken together, will be equal to a Circle, whoſe Radius is a mean Proportional, between the Diameter* AE *and a tranſverſe Line* BE, *drawn from one Extremity of the Diameter* E, *to the end of the Side* AB *next to the other Extremity.* This we will thus demonſtrate by the help of ſpecious Arithmetick; ſince BN, HN are the right Sines of equal Arches, CK and GK whole Sines, *&c.* and the Lines BH, GC, *&c.* parallel: having drawn obliquely the tranſverſe Lines HC, GD, all the An-
p. 14. gles at H, C, G, D, *&c.* will be equal by *Conſect.* 1. *Def.* 11. and conſequently all the Triangles BNA, HNI, ICK, *&c.* are equiangular, both among themſelves, and to the △ABE;
p. 89. ſince the Angle at B is a right one, by *Conſect.* 1. *Prop.* 33. and the Angle at A common to the Triangles BNA, ABE.

Wherefore as $\left.\begin{array}{l}\text{BN to NA}\\ \text{or HN to NI}\end{array}\right\}$ ſo $\begin{array}{l}\text{CK to KI,}\\ \text{\& GK to KL.}\end{array}$

and $\left.\begin{array}{l}\text{DM to ML}\\ \text{FM to ME}\end{array}\right\}$ as EB to BA; and ſo by making BN, HN, DM, FM $= a$, CK and GK $= b$, EB $= c$, for NA, NI, ML and ME, you may rightly put *ca*, and for KI and KL, *eb*; and for AB, *ec*. Which being done, you may eaſily obtain the conical Surfaces of the inſcrib'd Solid, and the Area of a Circle, whoſe Radius ſhall be a mean Proportional between AE and EB; and it will be evidently manifeſt, that theſe two are equal. For, 1. (*for the conical Surfaces*) the Diameter of the Baſe BH $= 2a$, and the Side of the Cone AB $= ec$: Therefore (making here n the Name of the Ratio between the Diameter and Circumference) the Circumference will be $2na$, which multiplied by half the Side $\frac{1}{2}ec$, gives the Conical Surface $naec$, by *Conſect.* 4. *Definit.* 18. And ſince the Circumference BH is (as before) $2na$, and the Circumference CG $= 2nb$, half of the Sum $2na + 2nb$, viz. $na + nb$ is the *equated* Circumference: Which multiplied by the Side BC $= ec$, gives the Surface of the truncated Cone BHGC $= naec + nbec$, by *Conſect.* 5. of the aforeſaid *Def.* *Laſtly*, Since the Surface of the truncated Cone DFGC is equal to the Surface of the truncated Cone BHGC, and likewiſe the conical Surface EDF to the conical Surface ABH, by adding, you'll have the Sum of them all $4naec + 2nbec$.

p. 29.

ib.

2. (*For the Area of the Circle*) The Diameter AE is $= 4ea + 2eb$, and BE $= c$: the Rectangle under theſe is $= 4eac + 2ebc$ (being evidently equal to the Rectangle of all the tranſverſe Lines BH, CG, DF into the Side AB, as *Archimedes* propoſes the thing) = to the Square of the Radius of the Circle ſought; becauſe the Radius is a mean Proportional between AE and BE, and ſo equal to $\sqrt{4eac + 2ebc}$, ſo that the whole Diameter is $2\sqrt{4eac + 2ebc}$. Therefore the Circumference of this Circle will be $2n\sqrt{4eac + 2ebc}$, i. e. $\sqrt{16nneac + 8nnebc}$: which multiplied by half the Semidiameter, *i. e.* by $\frac{1}{2}\sqrt{4eac + 2ebc}$, i. e. $\sqrt{eac + \frac{1}{2}ebc}$ gives the Area of the Circle ſought $\sqrt{16nnaaeecc + 16nneeabcc + 4nneebbcc}$. But this Root extracted is $4naec + 2nbec$, equal to the abovesaid Sum of the conical Surfaces. Q. E. D.

Having thus demonſtrated the Lemma, we will eaſily demonſtrate with *Archimedes* (tho' not after his way) *That the Surface of any Sphere, is the Quadruple of the greateſt Circle in it*, as is already evident from the 3d *Conſect.* For ſince all the conical Surfaces of the inſcribed Solid taken together, by the preceding Lemma, are equal to the Area of a Circle, whoſe Radius is a mean Proportional between the Diameter AE and the tranſverſe Line EB; and this mean Proportional approaches always ſo much nearer to the

Diameter

Diameter AE, and thoſe Surfaces ſo much nearer to the Surface of the Sphere, by how many the more Sides the inſcribed Figure is conceived to have, by *Conſect.* 1ſt and 2d, p. 28. *Def.* 18. if you conceive the Biſection of the Arches AB, BC, *&c.* to be continued *in infinitum*, it will neceſſarily follow, that all thoſe conical Surfaces will at length end in the Surface of the Sphere itſelf, and that mean Proportional in the Diameter AE; and ſo the Surface of the Sphere will be equal to a Circle, whoſe Radius is the Diameter AE. But that Circle would be Quadruple of the greateſt Circle p. 104. in the preſent Sphere, by *Prop.* 35. Therefore the Surface of the Sphere is the Quadruple of that Circle alſo. Q. E. D.

Hence alſo it will be very eaſy to deduce with *Archimedes* (tho' again after another way) that celebrated Propoſition, which we have already demonſtrated from another Principle in the *Prop.* of this *Schol.* viz. *That a Cylinder is to a Sphere of the ſame Diameter and Altitude, as* 3 *to* 2. For by putting *a* for the Diameter and Altitude, and *ea* for the Circumference, the Area of the circular Baſe will be $\frac{1}{4}eaa$; and this Area being multiplied by the Altitude *a*, gives $\frac{1}{4}ea^3$ p. 25. for the Content of the Cylinder, by *Conſect.* 5. *Definit.* 16. and four times the ſame, *i. e.* *eaa* multiplied by $\frac{1}{6}a$, gives 31. $\frac{1}{6}ea^3$ for the Content of the Sphere, by *Conſect.* 1. *Definit.* 20. 27. and *Conſect.* 3. *Definit.* 17. Wherefore the Cylinder will be to the Sphere as $\frac{1}{4}$ to $\frac{1}{6}$, *i. e.* in the ſame Denominator, as $\frac{6}{24}$ to $\frac{4}{24}$, *i. e.* as 6 to 4, or 3 to 2. Q. E. D.

Whence it is evident, that the Dimenſion of the Sphere would be every way abſolute, if the Proportion of the Diameter to the Circumference were known; which now with *Archimedes* we will endeavour to inveſtigate.

PROPOSITION XL.

THE Proportion of the Periphery of a Circle * *to the Diameter, is leſs than* $3\frac{1}{7}$ *or* $\frac{22}{7}$ *to* 1. *and greater than* $3\frac{10}{71}$ *to* 1.

* Archim. Cyclom. Prop. 2.

Demonſtration.

The whole Force of this Propoſition conſiſts in theſe Verities; 1. That any Figure circumſcribed about a Circle, has a greater Perimeter or Circuit than the Circle, but any inſcribed one, a leſs. 2. The Perimeter of a circumſcribed Figure of 96 Sides, has a leſs Proportion to the Diameter, than $3\frac{1}{7}$ to 1. and of an inſcribed Figure of the ſame Number of Sides, a greater than $3\frac{10}{71}$ to 1.

To demonſtrate this, we will enquire, 3. The Proportion of one Side of ſuch a Figure, whether circumſcribed or inſcribed, after the following way.

For the firſt Part of the Propoſition.

Suppoſe the Arch BC (*Fig.* 84. N. 1.) of 30 Degrees, and its Tangent BC forming a right Angle with the Radius AC, to make the Triangle ABC half of an equilateral one, ſo that AB ſhall be to BC in a double Ratio, *viz.* as 1000 to 500; which being ſuppoſed, AC will be the Root of the Difference of the Squares of BC and AB, *i. e.* a little greater than 866, but not quite $\frac{1}{100}$. p. 124.

Now, continually biſecting the Angles BAC by AG, GAC by AH, HAC by AK, KAC by AL, then as BC is half the Side of a circumſcribed Hexagon, GC is the half Side of a Dodecagon (or twelve-ſided Figure) HC of a Polygon of twenty-four Sides, KC of one of forty-eight. *Laſtly*, LC of one of ninety-ſix Sides; and by *N.* 3. *Schol.* 2. *Prop.* 34. GC will be to AC, as BC to BA + AC, and alſo HC to AC, as GC to GA + AC, *&c.* Wherefore, 93.

In the firſt Biſection, if GC be 500, then will AC be 1866, and a little more; and AG (which is the Root of the Sum of the Squares of GC and AG) $1931\frac{8}{10}$ +.

In the ſecond Biſection, if HC be 500, then will AC be found to be $3797\frac{8}{10}$ +, and AH $3830\frac{5}{10}$ +.

In the third Biſection, if KC be 500, then will AC be $7628\frac{1}{10}$ +, and AK $7644\frac{8}{10}$ +.

In the fourth Biſection, if LC be 500, then will AC be $15272\frac{9}{10}$ +.

Therefore LC taken 96 times, will give 48000 the Semi-Perimeter of the Polygon, which has the ſame Proportion to the Semi-Diameter AC $15272\frac{9}{10}$, as the whole Periphery to the whole Diameter. But 48000 contains $15272\frac{9}{10}$ three times, and moreover $2181\frac{3}{10}$ remaining Parts, which are leſs than $\frac{1}{7}$ Part of the Diviſion; for being multiplied by 7, they give only $1526\frac{1}{10}$ +.

Therefore it is evident, that the Periphery of this Polygon (and much more the Periphery of a Circle which is leſs than that) will have a leſs Proportion to the Diameter, than $3\frac{1}{7}$ to 1. Which is one thing we were to demonſtrate.

For the ſecond Part of the Prop.

Suppoſe the Arch BC to be (N° 2.) of 60 Deg. that is, the Angle BAC at the Periphery of 30 Deg. Then ſince

p. 89. the Angle at B is a right one by *Conſect.* 1. *Prop.* 33. the Triangle ABC will be again half an equilateral one, and BC the whole Side of an Hexagon, and GC of a Dodecagon, &c. So that ſuppoſing BC to be 1000 (as before we put 500 for the Side of the Hexagon) AC will be 2000, and AB the Root of the Difference of the Squares of BC and AC will be leſs than 1732$\frac{1}{10}$, *viz.* 1732, and not quite $\frac{5}{100}$.

Again biſecting continually the Angles BAC, GAC, &c. ſince the Angles at the Periphery BAG, GAC, GCB, ſtanding 89. on equal Arches BG and GC, are equal by *Prop.* 33. and the Angle at G (common to the Triangles GCF and GCA) and the others at H, K, L, are all right ones, by ib. *Conſect.* 1. of the afore-cited *Prop.* Theſe two Triangles CGF and CGA are equiangular; and conſequently, by *Prop.* 34. the Perpendicular GC in the one, will be to the Perpendicular GA in the other, as the Hypothenuſe CF in the one, to the Hypoth. AC in the other; *i.e.* (by the Foundation we have laid in the former Part of the Demonſtration of N° 3. *Schol.* 2. *Prop.* 34.) as BC to AB + AC; and in like manner, HC will be to HA, as GC to AG + AC, &c. Wherefore,

In the firſt Biſection, if GC be 1000, then AG will be a little leſs than 3732$\frac{1}{10}$; and AC (which is the Root of the Sum of the Squares of AG and GC) will be a little leſs than 3863$\frac{8}{10}$.

In the ſecond Biſection, if HC be 1000, then will AH be a little leſs than 7595$\frac{9}{10}$, and AC a little leſs than 7661$\frac{5}{10}$.

In the third Biſection, if KC be 1000, then will AK be a little leſs than 15257$\frac{4}{10}$, and AC a little leſs than 15290$\frac{2}{10}$.

In the fourth Biſection, if LC be 1000, then will AL be a little leſs than 30547$\frac{6}{10}$, and AC a little leſs than 30564; and conſequently, if LC be ſuppoſed 500, AC will be leſs than 15282.

Therefore LC taken 96 times, will give 48000 for the Perimeter of the inſcribed Polygon, and 15282, and a little leſs will be the Diameter AC. But 48000 contains 15282 thrice, and moreover a Remainder of 2154 Parts, which are more than $\frac{10}{71}$ of the Diviſor; for $\frac{1}{71}$ of this Number makes 215$\frac{17}{71}$ and ſo $\frac{10}{71}$ makes 2150$\frac{170}{71}$, *i. e.* 2152$\frac{28}{71}$. Therefore it is evident, that the Periphery of this inſcribed Polygon, (and much more the Periphery of the Circle which is greater than that) will be to its Diameter in a greater Proportion than $3\frac{1}{71}$ to 1, which is the ſecond thing to be proved.

SCHOLIUM.

IF any one had rather make uſe of the ſmall Numbers of *Archimedes*, which he choſe for this Purpoſe, by putting 306 in the firſt Part of the Demonſtration for AB, and 153 for BC; in the ſecond, 1560 for AC, and 780 for BC; by the like Proceſs of Demonſtration, he may infer the ſame with *Archimedes*. We like our Numbers beſt, tho' ſomewhat large, becauſe they may be remember'd, and are more proportionate to things, and make alſo the latter part of our Demonſtration like the former. In the mean while, the Proportion of the Diameter to the Periphery of the Circle, by the *Archimedean* way, is included within ſuch narrow Limits, that they only differ from one another $\frac{1}{497}$ or $\frac{10}{4970}$ Parts; for $\frac{10}{71}$ ſubſtracted from $\frac{10}{70}$, leaves $\frac{1}{497}$, as $3\frac{10}{71}$ or $\frac{223}{71}$, and $3\frac{10}{70}$ or $\frac{220}{70}$, if they are reduced to the ſame Denomination, make on the one hand $\frac{15610}{4970}$, and on the other, $\frac{15620}{4970}$. Hence, by dividing the Difference $\frac{10}{4970}$ into two Parts, it will be eaſy to expreſs a middle Proportion of the Periphery to the Diameter between the two *Archimedean* extreme ones, in theſe Numbers, *viz.* as 15615 to 4970 (or dividing both Sides by 5) as 3123 to 994, or (dividing both Sides by 7) as $446\frac{1}{7}$ to 142, or (by dividing again by 2) as $223\frac{1}{14}$ to 71, &c. which are as fit for uſe as thoſe of *Archimedes*, which we therefore uſe before any others, particularly in Dimenſions that do not require any great Exactneſs; but if they do, thoſe of *Ptolemy*, *Vieta*, *Ludolphus a Ceulen*, *Metius*, *Snellius*, *Lansbergius*, *Hugens*, &c. may be uſed: as if

The Diameter be . . . . The Circumference will be
10, 000, 000 . . . . 31, 416, 666. *Ptolem.*
10, 000, 000, 000 . . . . 31, 415, 926, 535. *Vieta.*
100, 000, 000, 000, 000, 000, 000 . . . . 314, 159, 265, 358, 979, 323, $846\frac{1}{2}$, &c. *Ludolph. a Ceulen.*

PROPOSITION XLI.

THE Area of a Circle has the ſame Proportion to the Square of its Diameter, as the fourth Part of the Circumference to the Diameter.

Demonſtration.

Though we have demonſtrated this before in *Conſect.* 6. *Prop.* 39. yet here we will do the ſame after another way. Since therefore the Circumference is a little leſs than $3\frac{1}{7}$, and a little more than $3\frac{10}{71}$ times the Diameter, putting z for this Exceſs, if the Diameter be 1, and the Circumference will be $3+z$; therefore the fourth Part of it will be $\frac{3+z}{4}$. But the Area of the Circle (by multiplying it by half the Semidiameter) *i. e.* $\frac{1}{4}$ the Diameter by the Circumference, is likewiſe $\frac{3+z}{4}$, and the Square of the Diameter is 1. Q. E. D.

CONSECTARY.

* *Archim. Prop.* 3. *Cyclom.*

THerefore if the * Proportion of *Archimedes*'s of 22 to 7, be near enough Truth to be made uſe of, the Area of the Circle will be to the Square of the Diameter as 11 to 14, becauſe the fourth Part of 22, *i. e.* $5\frac{1}{2}$, or $\frac{11}{2}$ to the Diam. 7, or $\frac{14}{2}$ is in the ſame Proportion.

PROPOSITION XLII.

† *Eucl. laſt Prop. lib.* x.

THE Diameter AC † *of a Square* (Fig. 83.) *is incommenſurable to the Side* AB (*and conſequently alſo to the whole Perimeter thereof*) i. e. *it bears a Proportion to it that cannot be exactly expreſſed by Numbers.*

Demonſtration.

For if you put 1 for AB, BC will be alſo 1; and by the *Pythagorick Theorem,* AC will be $=\sqrt{2}$. Therefore by *Conſect.* 4. *Definit.* 30. AC is incommenſurable to the Side AB, &c. Q. E. D. p. 47.

CONSECTARY I.

IT is notwithſtanding commenſurable in Power, for its Square is to the Square of the Side as 2 to 1.

CONSECTARY II.

NOW if the Proportion of the Side, or of the whole Perimeter ABCDA, to the Diam. AC is to be expreſſed meerly by Numbers, as we have done in the Diameter and Circumference of a Circle; then making the Side AB=100, the Diameter is greater than $141\frac{42}{100}$, and leſs than $141\frac{43}{100}$.

PROPOSITION XLIII.

THE Area of a Circle is incommenſurable to the Square of the Diameter.

Demonſtration.

For if the Semidiameter CD be divided (*Fig.* 85.) into two equal Parts (and conſequently the Diameter DF into four) p. 119.
AC will be 2, *viz.* $\surd 4$ and 𝔄ℭ, $\surd 3$, by *Schol.* 2. *Prop.* 34. 93.
N. 3. the Sum is $\surd 4 + \surd 3$, and the Sum of as many equal to the greateſt AC is 4. The Parts of the Semidiameter being again biſected, and AC will $=4$, or the $\surd 16$, *ac* $=\surd 15$, 𝔄ℭ $=\surd 12$, 𝔞𝔠 $=\surd 7$, the Sum is $\surd 16 + \surd 15 + \surd 12 + \surd 7$; and the Sum of as many equal to the greateſt AC is $=16$, &c. And thus the latter Sums will be the Square Numbers increaſing in a quadruple Proportion; but the former Sums will be always compoſed of the rational Root of every ſuch Square, and of ſeveral other irrational Roots of Numbers unevenly decreaſing; ſo that it will be impoſſible to expreſs thoſe former Sums by any rational Number, by what we have ſaid in *Schol.* 2. *Definit.* 30. 47.
Wherefore all the Indiviſibles of the Quadrant ADC, are to as many of the Square ACDE equal to the greateſt AC, *i. e.* the Quadrant ADC itſelf to the Square ACDE (and conſequently the whole Area of the Circle to this circumſcribed Square) will be as a Surd Quantity to a true Square Number; *i. e.* the Area of the Circle will be incommenſurable to the Square of the Diameter, by *Conſect.* 4. of the ſaid *Definit.* Q. E. D. 49.

CONSECTARY.

AND becauſe the fourth Part of the Circumference has the ſame Proportion to the Diameter, as the Area of the Circle to the Square of the Diameter, by *Prop.* 41. 117.
Therefore alſo that will be incommenſurable to this, and conſequently the whole Circumference will be ſo to the Diameter.

SCHOLIUM.

WHerefore it is ſomewhat wonderful, which *G. G. Leibnitius* * tells us, *viz.* That the Square of the Diameter being 1, the

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* *Acta Erudit. Ann.* 82. *p.* 44, &c.

Area of the Circle will be $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}$, *&c. ad infinitum*, (which Series is made by adding $1\frac{1}{5}-\frac{1}{3}$ and $+\frac{1}{9}-\frac{1}{7}$, *&c.*) to $\frac{2}{3}+\frac{2}{35}+\frac{2}{99}$, *&c. i.e.* to the Sum of infinite Fractions, whoſe common Numerator is 2. But their Denominators, every fourth Square leſſened by Unity, in the Series of the Squares of natural Numbers, which Sum might ſeem expreſſible in Numbers, ſince all its Parts are Fractions reducible to a common Denomination; while notwithſtanding *Leibnitius* himſelf confeſſes, that the Circle is not commenſurable to the Square, nor expreſſible by any Number.

CHAP. VII.

Of the Powers [*or Squares*] *of the Sides of Triangles, and other Regular Figures*, &c.

PROPOSITION XLIV.

p. 124.

* Eucl. 47. lib. I.

IN *right-angled Triangles* * (ABC, Fig. 86.) *the Square of the Side* (BC) *that ſubtends the Right-Angle, is equal to the Squares of the other Sides*(AB *and* AC) *taken together.*

Demonſtration.

Though we have demonſtrated this Truth more than once in the foregoing Part of this Book; yet here we will confirm it again as follows. Having deſcribed a Semicircle on each ſide of the Square BE, which will all neceſſarily touch one another in one Point, and be equal to the Semicircle BAC, if you conceive as many Triangles inſcribed alſo equal to BAC, it will be evident, that the Square BE will contain the ſaid four Triangles: and beſides the little Square FGHI, whoſe Side FI, *e. g.* is the Difference between CI, the greater Side of the Triangle, and CF the leſſer (for that the leſſer Side CF = BA, lying in the firſt Semicircle, if it be continued to I, in the ſecond Semicircle, makes CI = CA, the greater Side of the other Triangle, and ſo of the others, appears from hence, That as the Angles ABC, and ACB together make one right one; ſo likewiſe BCF (= CBA) and ECF make alſo one right one; and conſequently ECF is = ACB, and the Arch and the Line EI =

to the Arch and the Line AB, &c.) Wherefore, if the greateſt Side BC or BD of the given Triangle, &c. be called a, and AC, b, and the leaſt AC, or CF, &c. be called c; the □ of the Side BC will be $=aa$, and the Area of each Triangle $\frac{1}{2}bc$; and ſo the four Triangles together $2bc$: but the Side of the little Square in the middle, will be $b-c$, and its Square $bb+cc-2bc$: Wherefore, if to this you add the four Triangles, the Sum of all, *i.e.* the whole Square BE, will be $bb+cc=aa$. Q.E.D.

CONSECTARIES.

I. HEnce the Sides $AC=b$, and $AB=c$, that comprehend the right Angle, being given, the Hypothenuſe or Baſe that ſubtends the right Angle BC will be $=\sqrt{bb+cc}$.

II. But if $BC=a$, and $AC=b$ be given, and you are to find $AB=x$; becauſe $xx+bb=aa$, you'll have (taking away from both Sides bb) $xx=aa-bb$: therefore x, *i.e.* $AB=\sqrt{aa-bb}$.

III. If two right-angled Triangles have their Hypothenuſes and one Leg equal, the other will alſo be equal.

PROPOSITION XLV.

IN obtuſe-angled Triangles (Fig. 87. N. 1.) the Square of the Baſe or greateſt Side BC *ſubtending the obtuſe Angle* BAC, *is equal to the Squares of* * *the other two Sides* (AB *and* AC) *taken together; and alſo to two Rectangles* (CAD) *made by one of the Sides* (AC) *which contains the obtuſe Angle and its Continuation* AD *to the Perpendicular* BD, *let fall from the other Side.* p. 124.

* Eucl. Prop. 12. lib. ii.

Demonſtration.

Make $BC=a$, $AB=c$, $AC=b$, $AD=x$, then CD will be $=b+x$. Therefore □ of $BD=cc-xx$, by *Conſect.* 2. of the preceding *Prop.* In like manner, if □ of $CD=bb+2bx+xx$ be ſubſtracted from the □ of $BC=aa$, you'll have $aa-bb-2bx-xx=$ to the ſame □ of BD. Therefore, p. 118.

$$cc-xx=aa-bb-2bx-xx,$$

i.e. (adding xx on both Sides)

$$cc=aa-bb-2bx,$$

i.e. (adding bb and $2bx$ on both Sides)

$$cc+bb+2bx=aa.$$ Q.E.D.

CON-

CONSECTARY.

IN this laſt Equation, if you ſubſtract $cc + bb$ from both Sides, then will $2bx = aa - bb - cc$; and (if you moreover divide both Sides by $2b$) you'll have $x = \frac{aa - bb - cc}{2b}$ Which is the Rule, when you have the Sides of an obtuſe-angled Triangle given, to find the Segment AD, and conſequently the Perpendicular BD.

PROPOSITION XLVI.

IN acute-angled Triangles, * *the Square of any Side* (e.g. BC, Fig. 87. N. 2.) *ſubtending any of the Angles, as* A, *is equal to the Squares of the other two Sides* (AB *and* AC) *taken together, leſſened one by two Rectangles* (CAD) *made by one of the Sides, containing the acute Angle* (CA) *and its Segment* AD *reaching from the acute Angle* (A) *to the Perpendicular* (BD) *let fall from the other Side.*

Demonſtration.

Make again BC $= a$, AC $= b$, AB $= c$, AD $= x$; then will CD $= b - x$: Therefore $cc - xx =$ □ of BD, and $aa - bb + 2bx - xx$ (*i.e.* □ of BC − □ of CD) will alſo be = □ of BD.

Therefore $cc - xx = aa - bb + 2bx - xx$,
i. e. (adding xx to both Sides)
$cc = aa - bb + 2bx$,
i. e. (adding on both Sides bb, and ſubſtracting $2bx$)
$cc + bb - 2bx = aa$. Q. E. D.

CONSECTARIES.

I. IF in the laſt Equation, except one, you add bb on both Sides, and ſubſtract aa, you'll have $cc + bb - aa = 2bx$; and if moreover you divide both Sides by $2b$, you will have $\frac{cc + bb - aa}{2b} = x$: Which is the Rule, by having three Sides given in an acute-angled Triangle, to find the Segment AD, and conſequently the Perpendicular BD.

Therefore knowing the Segments AD and CD, and alſo the Perpendicular BD in oblique-angled Triangles, whether obtuſe-angled, or acute-angled; when moreover the Sides

---

* Eucl. Prop. 13. l. ii.

BC and AB are likewiſe given, the Angles of either right-angled Triangles, or oblique-angled ones, will be known; ſo that the laſt Caſe of Plain Trigonometry, which we deferred from *Prop.* 34. to this Place, may hence receive its Solution.

PROPOSITION XLVII.

THE Square (Fig. 88.) *of the Tangent* BD *to a* * *Circle, is equal to a Rectangle contained under the whole Secant* DA; *and that Part* DE *of it, which is without the Circle, whether the Secant paſſes thro' the Center or not.*

* *Eucl.* 36. *l.* iii.

Demonſtration.

For in the firſt Caſe, if CB or CE be $= b$, DE $= x$; then will CD $= b + x$, and AD $= 2b + x$: Therefore, ▭ ADE $= 2bx + xx$, and □ of CD $= bb + 2bx + xx$. Therefore, if from the □ of CD you ſubſtract □ of CB $= bb$, the Remainder will be $2bx + xx =$ □ BD $=$ ▭ ADE. *Q. E. D.*

In the ſecond Caſe, the Lines remaining as before, make DE $= y$, FE or FA $= Z$; therefore the ▭ ADE will be $= 2zy + yy$: but the □ of FC is equal to the □ of EC — □ of FE $= bb - zz$; and by the ſame reaſon, the □ of FD (= □ of CD — □ of FC) will be $= bb + 2bx + xx - bb + zz$, i. e. $2bx + xx + zz$: but the ſame Square of FD is $= zz + 2zy + yy$. Wherefore taking away from theſe equal Squares the common one zz, you'll have $2zy + yy = 2bx + xx$, *i. e*, by the 1ſt *Caſe* = □ of BD. *Q. E. D.*

CONSECTARIES.

I. THerefore the Rectangles of divers Secants (as of ADE and *ade*, in *Fig.* 88. *N.* 1.) which are equal to the ſame Square of BD, are equal alſo to one another; which the laſt Equation in our Demonſtration ($2zy + yy = 2bx + xx$) is an ocular Proof of.

II. Therefore, by *Prop.* 19. as *a*D to AD, ſo reciprocally is DE to D*e* in the *Fig.* of the ſecond Caſe.

III. Tangents to the ſame Circle drawn from the ſame Point, as DB and D*b* (*N.* 1.) are equal; becauſe the Square of each is equal to the ſame Rectangle.

IV. Nor can there be more Tangents drawn from the ſame Point than two; for if beſides DB and D*b*, D*c* could alſo touch the Circle, then it would be equal to them, by *Conſ.* 2. but that is abſurd, by *Conſ.* 3. *Def.* 7.

SCHOLIUM I.

HENCE is evident the Original of the Geometrical Conſtructions which *Des Cartes* makes uſe of, *p.* 6, & 7. in reſolving theſe three Equations, $ZZ = aZ + bb$, $ZZ = -aZ + bb$, and $ZZ = aZ - bb$. For in the firſt Caſe, ſince he makes NO or NL (*Fig.* 89. *N.* 1.) or $NP = \frac{1}{2}a$, and the Tangent $LM = b$, he ſays, that MNO drawn thro' the Center, will be $= Z$, the Quantity ſought; which thus appears: Making $MO = Z$, then will $NM = Z - \frac{1}{2}a$, and its □ will be $ZZ - aZ + \frac{1}{4}aa$. But the Rectangle OMP, (which is by the preſent Prop. = □ of LM, i. e. bb) together with the □ of NL or NP (i. e. $\frac{1}{4}aa$) is = □ of NM by the *Pythagor. Theor.* Therefore $ZZ - aZ + \frac{1}{4}aa = bb + \frac{1}{4}aa$. i. e. (taking away from both ſides $\frac{1}{4}aa$) $ZZ - aZ = bb$. i. e. (by adding on both ſides aZ) $ZZ = aZ + bb$; which is the Equation propoſed. In the ſecond Caſe, if you make $PM = Z$ (as *Des Cartes* makes it) you'll have □ of $NM = ZZ + aZ + \frac{1}{4}aa = bb + \frac{1}{4}aa$; as before. Therefore $ZZ + aZ = bb$; and ſo $ZZ = -aZ + bb$: which is the Equation of the ſecond Caſe. In the third Caſe, whether you make the whole Secant RM (*N.* 2.) or that part QM of it without the Circle $= Z$ the Root ſought, there will come out on both ſides the ſame Equation of the third Caſe: and ſo it is manifeſt, that this Equation has thoſe two Roots. For if RM be $= Z$ (adding moreover to the Fig. of *Des Cartes* the Line NO biſecting QR, and ſo making OM = LN) OR or OQ will be $= Z - \frac{1}{2}a$, and ſo the □ of $OQ = ZZ - aZ + \frac{1}{4}aa$; and this, together with the Rectangle RMQ (which is by virtue of the preſent Prop. = □ of LM) = □ of NQ, *i. e.* □ of OM, *i. e.* $ZZ - aZ + \frac{1}{4}aa + bb = \frac{1}{4}aa$, i. e. (by adding aZ, and taking away $\frac{1}{4}aa$) $ZZ + bb = aZ$; i. e. (taking away bb) $ZZ = aZ - bb$: which is the very Equation of the third Caſe. But if QM be made $= Z$, OQ or OR will $= \frac{1}{2}a - Z$, and its □ $= \frac{1}{4}aa - aZ + ZZ$, as well as the former; and ſo all the reſt. Q. E. D.

SCHOLIUM. II.

NOW if you would immediately deduce theſe Rules by the preſent Propoſition, without the *Pythag. Theor.* it may (*e. g.* in the firſt Caſe) be done much ſhorter thus:

If

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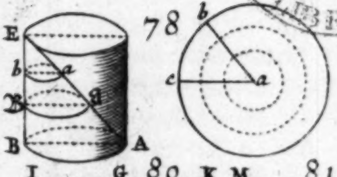

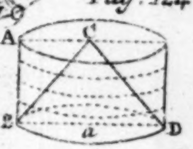

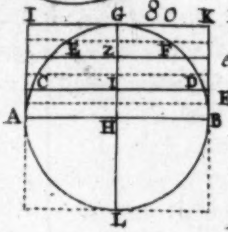

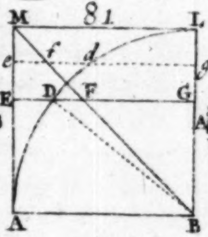

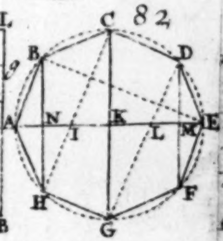

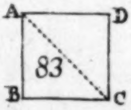

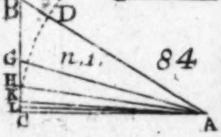

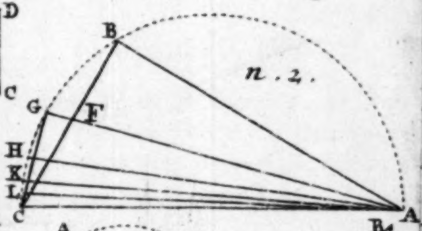

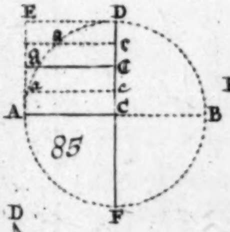

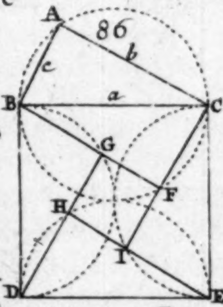

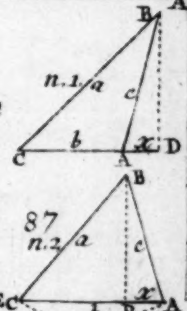

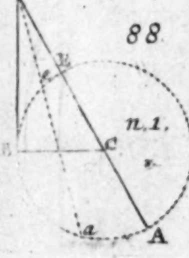

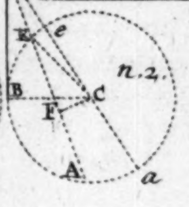

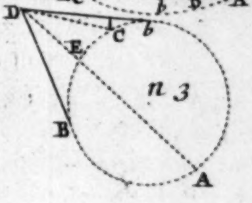

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If $MO = Z$, and NO or $NP = \frac{1}{2} a$, then will $PM = Z - a$; therefore the Rectangle $OMP = ZZ - aZ = bb$, or □ of LM, by the preſent Propoſition. Therefore by adding to both ſides aZ, you'll have $ZZ = aZ + bb$, which is the very Equation of the firſt Caſe. In the ſecond Caſe, if MP be $= Z$, and ſo $OMZ + a$; then ▭ $OMP = ZZ + aZ = bb$. Therefore $ZZ = - aZ + bb$. In the third Caſe, if $MR = Z$, then QM or $PR = a - Z$: therefore ▭ $RMP = aZ - ZZ = bb$, i. e. $aZ = bb + ZZ$, or $aZ - bb = ZZ$; but if $QM = Z$, RM will be $= a - Z$. Therefore the Rectangle $RMP = aZ - ZZ = bb$, as before. p. 123.

SCHOLIUM III.

FROM the 2d *Conſect.* of the preſent Propoſition flows another Rule for ſolving the laſt Caſe of *Plain Trigonometry*, which we ſolved in the *Conſect.* of the foregoing Propoſition; *viz.* If you have all the three Sides of the Oblique-angled Triangle BCD (*Fig.* 90) given, and from 133. the Center C, with the diſtance of the leſſer Side CB you deſcribe a Circle, then by *Conſect.* 2. of the preſent Propo- 123. ſition, BD the Baſe of the Triangle (which is the greateſt Side, or in an Equicrural Triangle, one of the greateſt) will be to AD, (the Sum of the Sides DC + CB) as DE the Difference of the Sides, to DF the Segment of the Baſe without the Circle: which being found, if the remainder of the Baſe within the Circle be divided into two equal parts, you'll have both FG and GB, as alſo DG, which being given; by help of the Right-angled Triangles GBC and GDC, all the Angles required may be found.

PROPOSITION XLVIII.

IN *any Quadrilateral Figure* * ABCD (Fig. 91. N. 1.) *inſcribed in a Circle, the Rectangle of the Diagonals* AC *and* BD *is equal to the two Rectangles of the oppoſite Sides* AB *into* CD, *and* AD *into* BC. 133.

* Ptol. lib. i. Almag.

Demonſtration.

Having drawn AE ſo that the Angle BAE be equal to the Angle CAD, then the Triangles ACD, ABE, will be ſimilar; for the other Angles EBA and ACD in the ſame Segment are equal, by virtue of *Conſect.* 1. *Prop.* 33. and 89. conſequently (by *Prop.* 34.) as AC to CD, ſo is AB to BE. 90. Wherefore by making $AC = a$, and $CD = ea$, and $AB = b$, then BE will be $= eb$. In like manner, the Triangles BAC and EAD are ſimilar; becauſe the Angles BAE, DAF, being equal by Conſtruction, if the common Angle EAF be added

added to each of them, the Angles BAF, and DAE, are equal; and beſides, the Angles BCA, EDA, in the ſame Segment likewiſe equal. Therefore as AD to DE, ſo is AC to CB: wherefore by putting, as before, a for AC, and oa for CB, and c for AD, then DE will $= oc$. Therefore the whole BD $= eb + oc$. Therefore the Rectangle of AC into BD will $= eba + oca =$ to the Rectangle of AB into CD $= eba$ ▭ $+$ of AD into BC $= oac$. Q. E. D.

SCHOLIUM.

IN Squares and Rectangles (*N.* 2.) the thing is ſelf-evident; for in Squares, if the Side be a, the Diagonals AC and BD will be $\sqrt{2aa}$, and ſo their Rectangle $= 2aa$ will be manifeſtly equal to the two Rectangles of the oppoſite Sides. In Oblongs, if two oppoſite Sides are a, and the two others b, the Diagonals will be $\sqrt{aa + bb}$, and their Rectangle $aa + bb$ is manifeſtly equal to the two Rectangles of the oppoſite Sides.

PROPOSITION XLIX.

* Eucl. 12. lib. xiii.

THE Side (AB) *of an Equilateral Triangle* (ABC, Fig. 92. N. 1.) *inſcribed in a* * *Circle, is in power triple of the Radius* (AD) i. e. *the* □ of AB *is triple of the* □ *of* AD.

Demonſtration.

MAKE AD or FD $= a$, and ſo its Square is aa. Then if DF be drawn thro' the middle of AB, or the middle of the Arch AFB, DE will be $= \frac{1}{2}a$; [for the Angles at E are right ones, by *Conſect.* 5. *Definit.* 8. and the Hypothenuſes AD, AF, are equal, by the *Schol.* of *Def.* 15. but the Side AE is common. Therefore the other Sides FE and ED are equal, by *Conſect.* 3. *Prop.* 43.] and the □ of it $\frac{1}{4}aa$, which, ſubſtracted from aa, leaves $\frac{3}{4}aa$ for the □ of AE. Therefore the Line AE is $\sqrt{\frac{3}{4}aa}$, and conſequently AB $2\sqrt{\frac{3}{4}aa}$, *i. e.* $\sqrt{\frac{12}{4}aa}$, *i. e.* $\sqrt{3aa}$: therefore □ AB $= 3aa$. Q. E. D.

CONSECTARIES.

1. IF the Radius of a Circle be $= a$, the Side of an inſcribed regular Triangle will be $\sqrt{3aa}$, *e.g.* if AD be 10, AB will be $\sqrt{300}$; and if AD be 10,000,000, AB

will be $\sqrt{300,000,000,000,000}$, *i. e.* 17320508, and the Perpendicular DE 5000, 000.

II. Hence it is evident, in the geneſis of a *Tetrahedron* propoſed in *Def.* 22. that the Elevation CE (*Fig.* 44. N. 1.) is to CF the remaining Part of the Diameter of the Sphere, as 2 to 1; for making the Radius CB $= a$, and its □ aa, the Square of AB or BE will $= 3aa$, by the preſent Propoſition. Therefore the □ of CB being ſubſtracted from the □ of BD or BE, there remains the □ of CE $= 2aa$. But ſince CE, CB, CF, are continual Proportionals, by *N.* 3. *Schol.* 2. *Prop.* 34. CE will be to CF as the □ of CE to the □ of CB, by virtue of *Prop.* 35. *i. e.* as 2 to 1.

SCHOLIUM.

HENCE you have the *Euclidean* way of generating † a *Tetrahedron*, and inſcribing it in a given Sphere, when he bids you divide the Diameter EF of a given Sphere, ſo that EC ſhall be 2, and CF 1, and then at EF to erect the Perpendicular CA, and by means thereof to deſcribe the Circle ABD, and to inſcribe therein an Equilateral Triangle, *&c.*

† *Eucl. Prop.* 13. *lib.* xiii.

PROPOSITION L.

T*HE Side* (AB) *of a regular Tetragon* (*or Square*) (ABCD, Fig. 92. N. 2.) *is double in power of the Radius* AD.

Demonſtration.

For having drawn the Diameters AC and BD, the Triangle AOB is right-angled, and conſequently, by the *Pythagorick Theorem*, if the □ of AO and BO be made equal to aa, then will the □ of AB $= 2aa$. Q. E. D.

CONSECTARY.

THerefore when the Radius of the Circle AO is made $= a$, the Side of the □ AB will $= \sqrt{2aa}$, *e. g.* if AO be 10, AB will be $\sqrt{200}$; and if AO be 10,000,000, AB will be $\sqrt{200,000,000,000,000}$, *i. e.* 14142136.

PROPOSITION LI.

T*HE Side* AB *of a regular Pentagon* * (ABCDE) (Fig. 93. N. 1.) *is equal in power to the Side of an Hexagon and Decagon inſcribed in the*

* *Eucl. Prop.* 10. *lib.* xiii.

the ſame Circle, i. e. *the* □ *of* AB *is equal to the Squares of* AF *and* AO *taken together.*

Demonſtration.

Make AO $=a$, AF $=b$, and AB $=x$: We are to demonſtrate that $xx=aa+bb$; which may be done by finding the Side AB by the Parts BH and HA, after the following way: Firſt, The Angle AOB is 72°, and the others in that Triangle at A and B, are each 54°. But BOG is alſo 54°, as ſubtending the Decagonal Arch BF of 36°, and FG of 18° alſo one half of it. Therefore the Triangles ABO and HBO are Equiangular, whence

$$AB : BO :: BO : BH$$

$$x : a :: a : \frac{aa}{x}$$

Secondly, in the Triangle BFA, the Angles at B and A are equal, by *N.* 3 *Conſect.* 5. *Def.* 8. and by virtue of the ſame, alſo the Angles at F and A, in the △ FHA are ſo too. Wherefore the Triangles BFA and FHA are Equiangular. Therefore

$$BA : AF :: AF : AH$$

$$x : b :: b : \frac{bb}{x}$$

Whence the whole Line AB will be $=\frac{aa+bb}{x}$; (becauſe the Part AH is found $=\frac{bb}{x}$ and BH $=\frac{aa}{x}$) and ſo $\frac{aa+bb}{x}$ is $=x$, and, by multiplying both Sides by x, $aa+bb=xx$. Q. E. D.

CONSECTARY I.

THerefore if the Radius of a Circle be called a, the Side of a Pentagon AB will be $\sqrt{aa+bb}$.

CONSECTARY II.

THerefore the □ of AI $=\frac{aa+bb}{4}$, and □ of OI = □ of OA − □ of AI $=aa-\frac{aa-bb}{4}$, i. e. $\frac{3aa-bb}{4}$. Therefore OI $=\sqrt{3aa-bb}$: Which yet may be expreſſed otherwiſe, *viz.* OI $=\frac{a+b}{2}$.

Demonſtration.

Make * OA or OF (*N.*2.) as before $=a$, AF $=b$, and FI now $=x$; then will OI $=a-x$ and having drawn the Arch FK with the Interval AF, ſo that AK may be equal to AF, and FI $=$ IK $=x$; then will the Angle IKA $=$ F $=72°$. Therefore the Angle AKO $=108°$; and ſince KOA is 36°, KAO will be alſo 36°, and ſo KA $=$ KO $=$ AF $=b$. Wherefore OI is $=b+x$, which was above $a-x$. Therefore 2OI $=a-x+b+x$, i. e. $a+b$. Therefore OI $=\frac{a+b}{2}$.

* *Eucl. Prop.* 1. *lib.* xiv.

CONSECTARY III.

THerefore the difference between the Perpendicular of the Triangle DE (*Fig.* 92. *N.* 1.) and the Perpendicular of the Pentagon OI is $=\frac{1}{2}b$, by virtue of *Conſect.* 2. of this, and of the Demonſtrat. of *Prop.* 49.

CONSECTARY IV.

HENCE is alſo evident, by the preſent Propoſition, that which in *Faber*'s Geneſis of an *Icoſahedr. Def.* 22. we ſaid, *viz.* that (ſee *Fig.* 46.) B*a* is equal to the ſide of a Pentagon BA, becauſe F*a* is $=$ to the Semidiameter OB, and BF is the ſide of a Decagon.

PROPOSITION LII.

THE ſide of a Hexagon is in power equal to the Radius, as being itſelf equal to it by N. 1. Schol. Def. 15.

PROPOSITION LIII.

THE ſide of a regular Octagon ABCD, *&c.* (Fig. 94.) *is equal in power to half the ſide of the Square, and the difference* (PB) *between that half-ſide and the Radius, taken together.*

Demonſtration.

That the □ of AB is $=$ to the □ of AP $+$ □ of BP, is evident from *Pythag. Theor.* But that PO is $=$ PA, half the ſide of the Square, is evident from the Equality of the Angles PAO and POA, ſince each is half a right one, or 45°. Wherefore the ſide of the Octagon is equal in power to half the ſide of the Square, *&c.* Q. E. D.

CONSECTARY.

THerefore, if the Radius be $=a$, then AP, by virtue of the *Pythag. Theor.* will be $=\sqrt{\frac{1}{2}aa}$, and PB $=a-\sqrt{\frac{1}{2}aa}$: the Square of the former is $\frac{1}{2}aa$, and of the latter $1\frac{1}{2}aa-\sqrt{2a^4}$. Therefore the Sum of theſe Squares, that is, the □ of AB, the ſide of the Octagon is $2aa-\sqrt{2aa^4}$, and the ſide of the Octagon is $\sqrt{2aa-\sqrt{2a^4}}$. For example, If AO be 10, AB will be $\sqrt{200-\sqrt{20000}}$, and if the Radius AO be 10000000, AB will be $\sqrt{200,000,000,000,000-\sqrt{20000000000000000000000000000}}$, by virtue of the preſent *Conſect.* or from the *Prop.* itſelf, if AO be 10000000, AP, by virtue of the *Conſ.* of *Prop.* 50. will be $=7071068$, and conſequently BP $=2928932$. The Squares of theſe added together give the □ of AB, and the Root thence extracted will be AB $=7653668$.

PROPOSITION LIV.

THE ſide of a regular Decagon * *is equal in Power to the greateſt part of the ſide of an Hexagon cut into mean and extreme Proportion.*

* *Eucl.* 9. *lib.* xiii. *Corol.*

Demonſtration.

Suppoſe BD (*Fig.* 95.) divided in mean and extreme Proportion in E, and BA to be joined to it length-ways = to the ſide of a Decagon inſcribed in the ſame Circle, whoſe Radius is BC or AC = BD. Now we are to demonſtrate that DE, the greateſt part of the ſide BD of the Hexagon divided in mean and extreme Proportion, is equal to BA, the ſide of a Decagon, and the Power of the one equal to the Power of the other. Becauſe the Angle ACB is 36°, ABC and A 72°, and conſequently CBD 108°, BCD and D will be each 36°, and ſo the whole ACD 72°. (ſo that CD will paſs preciſely thro the other end of the ſide of the Pentagon AF.) Wherefore the Triangles ABC and ADC are Equiangular, and

$\begin{matrix}\text{AD to AC}\\ \textit{i.e.}\ \text{BD}\end{matrix}\left\{\begin{matrix}\text{as AC}\\ \textit{i.e.}\ \text{BD}\end{matrix}\right\}$ to AB. Therefore the whole Line AD is divided in mean and extreme Proportion. But BD is alſo divided in the ſame Proportion by the Hypoth. Wherefore

As AD to DB, ſo is DB to BA;
So is DB to DE, and DE to EB.

Therefore DB is in the ſame Proportion to DE as DB to BA. Therefore DE is = BA, and the Power of the one to the Power of the other. Q. E. D.

CONSECTARIES.

THerefore, if the Radius or the ſide of the Hexagon be *a*, the ſide of the Decagon will be $\sqrt{\frac{5}{4}aa}-\frac{1}{2}a$, by *Schol.* 2. *Prop.* 27. *e.g.* if the Radius be 10, the ſide of the Decagon will be $\sqrt{125}-5$, and if the Radius be 10,000,000, the ſide of the Decagon will be found $=\sqrt{125,000,000,000,000}$ $-5000,000$, *viz.* by adding the Square of the Radius and the Square of half the Radius into one Sum, and from the Root thereof ſubſtracting half of the Radius; whence you will have the ſide of the Decagon = 6180340, the half whereof 3090170 gives the difference between the Perpendiculars of the Triangle and the Pentagon, by *Conſect.* 3. *Prop.* 51.

II. Therefore the ſide of the Pentagon, by *Prop.* 51. is $=\sqrt{\frac{5}{2}aa-\sqrt{\frac{5}{4}a^4}}$; for the Square of the Hexagon is aa or $\frac{2}{2}aa$; and the □ of the Decagon $\frac{3}{2}aa-\sqrt{\frac{5}{4}a^4}$: whence the Root is extracted out of the Sum of theſe is the ſide of the Pentagon, *viz.* $\sqrt{\frac{5}{2}aa-\sqrt{\frac{5}{4}a^4}}$; *e. g.* if the Radius be 10, the ſide of the Pentagon will be $\sqrt{250\sqrt{125000}}$, and if the Radius be 10,000,000, ſince the ſide of the Hexagon is equal to it, and the ſide of the Decagon 6180340, their Squares being added into one Sum, the Root extracted out of that Sum will give the ſide of the Pentagon 11755704 nearly; and the ſides being collected into one Sum, the half of it 8090170 will give the Perpendicular in the Pentaglon OI, by *Conſect.* 2. *Prop.* 51.

SCHOLIUM II.

TO illuſtrate what we have deduced in the Conſectaries of *Prop.* 51. it may be proper to obſerve, that if *a* be put $=10$ or $\frac{1000}{100}$, the ſide of the Decagon will be $=\sqrt{125}-5$, *i.e.* $\frac{618}{100}$ nearly $=b$; therefore the □ $aa=\frac{1000000}{10000}$ and the □ $bb=\frac{381924}{10000}$: therefore $aa+bb$ or the □ of AB $=\frac{1381924}{10000}$, the Perpendicular OI $=\frac{a+b}{2}=\frac{1618}{100}$

divided by 2, that is, $\frac{809}{100}$. Now the □ of AI is $\frac{1}{4}$ of the □ of AB $= \frac{345481}{10000}$, and the □ of OI $= \frac{aa + 2ab + bb}{4} = \frac{654481}{10000}$. Now if you add the □ of AI and the □ of OI, the Sum will be = □ of AO $= \frac{2aa + 2ab + 2bb,}{4}$ *i. e.* $\frac{aa + ab + bb}{2} = \frac{99}{1}\big|\frac{996}{0000}$, or nearly 100. Thus likewiſe, ſince the Perpendicular OI, above found, in *Conſect.* 1. may be alſo determined by $\frac{\sqrt{3aa + bb}}{4}$, becauſe aa is $= \frac{1000000}{10000}$, and $3aa$ $\frac{3000000}{10000}$, ſubſtracting from it $bb = \frac{381974}{10000}$, and the Remainder will be $\frac{2618076}{10000}$, and this being divided by 4, you will have the □ of OI $= \frac{654519}{10000}$, and the Root of it extracted is $\frac{809}{100}$ nearly; ſo that thoſe two different Quantities in *Conſect.* 1. will rightly expreſs the ſame Perpendicular OI.

SCHOLIUM II.

THerefore as we have now practical Rules to determine Arithmetically the ſides of the Pentagon and Decagon, ſo alſo they may be found Geometrically by what we have demonſtrated. For if the Semidiameter CB (*Fig.* 96. *N.*1.) be divided into two equal parts, EC will be $= \frac{1}{2}a$; and erecting perpendicularly the Radius CD $= a$, DE will $= \sqrt{\frac{5}{4}aa}$. Moreover, if you cut off EF equal to ED, FC will be $= \sqrt{\frac{5}{4}aa} - \frac{1}{2}a =$ to the ſide of the Decagon, by *Conſect.* 1. Having therefore drawn DF, which is equal in Power to the Radius or Side of the Hexagon DC, and the ſide of the Decagon FC together, by the *Pythag. Th orem* it will be the ſide of the Pentagon ſought. Much to the ſame purpoſe is alſo this other new Conſtruction of the ſame Problem, wherein BG (*N.* 2.) is the ſide of the Hexagon, and BD the ſide of the Square, to which GF is made equal; ſo that FC is the ſide of the Decagon, and DF of the Pentagon, which we thus demonſtrate: Suppoſing GH, the ſide of an Equil. Triangle, the Square of GE will be $\frac{3aa}{4}$, by *Prop.* 48. which being ſubſtracted from the Square

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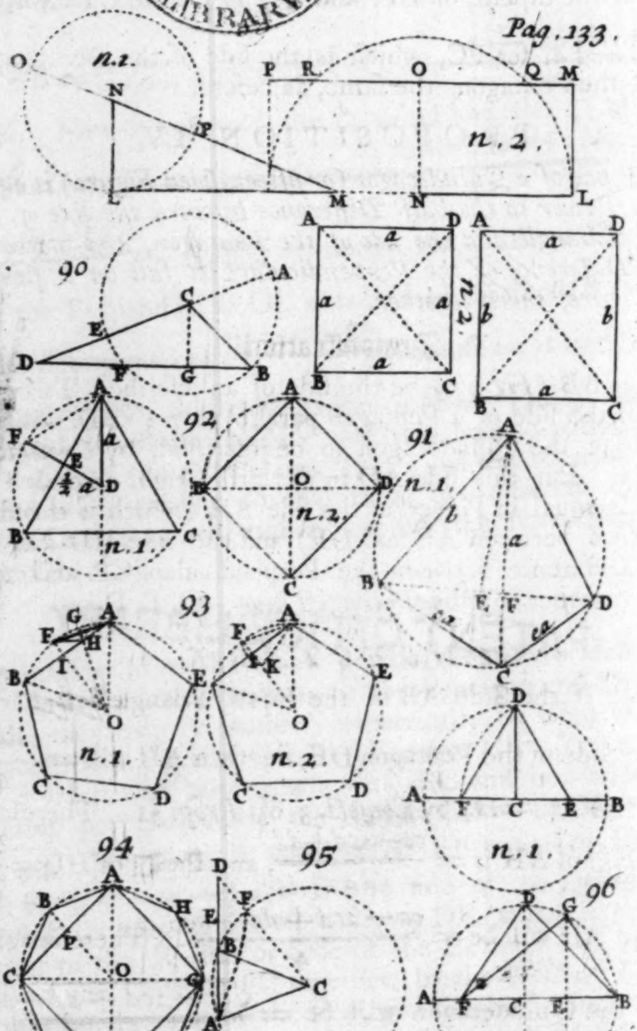

of GF $= 2aa$, *viz.* $\frac{3}{4}aa$, by *Prop.* 49. there will remain $\frac{5aa}{4}$ for the Square of EF, and $\sqrt{\frac{5}{4}aa}$ for the Line EF, and $\sqrt{\frac{5}{4}aa - \frac{1}{2}a}$, for FC, which is the ſide of the Decagon, as DF of the Pentagon, the ſame, as before.

PROPOSITION LV.

THE ſide of a Quindecagon (or fifteen-ſided Figure) is equal in Power to the half Difference between the ſide of the Equil. Triangle and the ſide of the Pentagon, and moreover to the Difference of the Perpendiculars let fall on a ſide of each Figure, taken together.

Demonſtration.

For if AB (*Fig.* 97.) be the ſide of an inſcribed Triangle, and DE the ſide of a Pentagon parallel to it; AD will be the ſide of the Quindecagon to be inſcribed, by *Conſect.* 4. *Def.* 15. But this ſide AD in the little right-angled Triangle, is equal in Power to the ſide AH (which is the half Difference between AB and DE) and the ſide HD (which is the Difference between the Perpendiculars CF and CG) taken together by the *Pythag. Theor.* Q. E. D.

CONSECTARY.

HEnce if the Side AB of the Equil. Triangle be called c, and the Side of the Pentagon DE, d; then AH will $= \frac{c-d}{2}$ while HD is $= \frac{1}{2}b$, by *Conſect.* 3 of *Prop.* 51. Therefore ſince the □ of AH is $= \frac{cc - 2cd + dd}{4}$, and the □ of HD $= \frac{bb}{4}$, the □ of AD will be $= \frac{cc - 2cd + dd + bb}{4}$. Therefore the Side of the Quindecagon will be $= \frac{\sqrt{cc - 2cd + dd + bb}}{4}$ that is, collecting the Square of the half Difference of the Sides of the Triangle and Pentagon, and the Square of the Difference of the Perpendiculars into one Sum; and then extracting the ſquare Root of that Sum, you will have the Side of the Quindecagon ſought. *E.g.* If the Radius CI be 10000000, the Difference of the Sides of the Triangle 17320508, and of the Side of the Pentagon 117557704, will be 5564804, and the half of this 2782402; but the Difference between the Perpendicular CF and the Perpendicular CG, is 3090170. The Squares therefore of theſe two laſt Numbers being collected into one Sum, and the Root extracted, will give the Side of the Quindecagon 4158234 nearly.

SCHOLIUM.

HERE we will ſhew the excellent Uſe of theſe laſt Propoſitions in making the Tables of Sines. For ſuppoſing the Radius to be 10000000, having found as above, the Sides of the chief regular Figures, if they are biſected, you will have ſo many primary Sines, *viz.* From the Side of the Triangle, you will get the Sine 8660254 of 60 Degrees; from the Side of the Square, the Sine 7071068 of 45°; from the Side of the Pentagon, the Sine 5877853 of 36°; from the Side of the Hexagon, the Sine 5000000 of 30°; from the Side of the Octagon, the Sine 3826834 of 22° 30′; from the Side of the Decagon, the Sine 3090170 of 18°; and, laſtly, from the Side of the Quindecagon, the Sine of 12° = 2079117. From theſe ſeven primary Sines, you may find afterwards the reſt, and conſequently all the Tangents and Secants according to the Rule we have deduc'd, *N.* 3. *Schol.* 5. *Prop.* 34. and which *Ph. Lansberg* illuſtrates in a prolix Example, in his Geom. of Triangles, *lib.* 2. *p.* 7, *&c.* But having found theſe greater Numbers of Sines, Tangents, *&c.* after what way Logarithms have been of late accommodated to them, remains now to be ſhewn, which in brief is thus: The Logarithms of the Sines, *&c.* might immediately be had from the Logarithms of Vulgar Numbers, if the Tables of Vulgar Numbers were extended ſo far, as to contain ſuch large Numbers: and thus, for example, the Logarithm of the Sine of 0° gr. 34′. which is 98900 in the Chiliads of *Ulacque* is 49951962916. But becauſe the other Sines, which are greater than this, are not to be found among the Numbers in the common Tables; (for they aſcend not beyond 100000, or others only reaching to 10000 or 20000) there is a way of finding the Logarithms of greater Numbers, than what are contained in the Tables. *e. g.*

If the Logarithm of the Sine of 45°, which is 7071068, is to be found, all this Number is not to be found in any common Tables, yet its firſt four Figures 7071 are to be found in the Tables of *Strauche*, with the correſpondent Logarithm 3.8494808; and the five firſt 70710 in the Tables

bles of *Vlacque*, with the Log. 4.8494808372. One of theſe Logarithms, *e.g.* the latter, is taken out of the Tables only by augmenting the Characteriſtick with ſo many Units, as there remain Figures in the Number propoſed, which are not found in the Tables, ſo that the Log. taken thus out will be 6.8494808372. Then multiplying the remaining Figures of the propoſed Number by the Difference of this Logarithm from the next following, (which for that purpoſe is every where added in the *Vlacquian* Chiliads, and is in this caſe 61419) and from the Product 4176492, caſt away as many Figures as adhere to the propoſed Number beyond the tabular ones, in this caſe two; for if the Remainder 41764 be added to the Logarithm before taken out, there will come out the Logarithm required, 6.8494850136, according to the Tables of *Vlacque*, wherein for the Log. of 10 you have 10,000,000,000; but according to thoſe of *Strauche*, which, for the Logarithm of 10, have only 10,000,000, you muſt cut off the three laſt Figures, that the Logarithm of the given Sine may be 6.8494850; as is found in the *Strauchian* and other Tables of Sines, except that inſtead of the Characteriſtick 6, there precedes the Characteriſtick 9, the reaſon whereof is this: If the Characteriſticks had been kept, as they were found by the Rule juſt now given, the Logarithm of the whole Sine (which in the *Strauchian* Tables is 10,000,000) would have come out 70,000,000, incongruous enough in Trigonometrical Operations. Wherefore that the Log. of the whole Sine might begin with 1, for the eaſineſs of Multiplication and Diviſion, there is aſſumed 100,000,000; the Characteriſtick being augmented by 3, wherewith it was conſequently neceſſary to augment alſo all the antecedent ones; and hence, *e.g.* the Logarithm of the leaſt Sine 2909, begins with the Characteriſtick 6, which otherwiſe, according to the Tables of common Numbers, would have been 3.

The Logarithms of all the Sines being found in this manner (tho' if you have found the Logarithms of the Sine of 45°, and moreover the Logarithm of the Sine of 30°, the Logarithms of all the reſt may be compendiouſly found by Addition and Subſtraction, from a new Principle which now we ſhall omit) the Logarithms of the Tangents and Secants may eaſily be found alſo, only by working logarithmically, according to the Rules of *Schol.* 5. *Prop.* 34. *N.* 5, and 6.

PROPOSITION LVI.

* Eucl. 13. lib. xiii.

*THE Side of a Tetrahedron * or equilateral Pyramid, is in power to the Diameter of the circumſcribed Sphere, as 2 to 3.*

Demonſtration.

For becauſe by the Geneſis of the *Tetrahedron*, Def. 22. (ſee *Fig.* 44. *N.* 1.) and *Schol. Prop.* 49. OC is $\frac{1}{3}$ of the Semidiameter OB, which we will call a; the □ of CB will be $=\frac{8}{9}aa$, by the *Pythag. Theor.* and ſo the Power (or Sq.) of the Side of the *Tetrahedron* will $=\frac{24}{9}aa$, by *Prop.* 49. but the Power of the Diameter $2a$, or $\frac{6}{3}a$, is $\frac{36}{9}aa$. Therefore the Power of the Side of the *Tetrahedron* is to the Power of the Diam. as 24 to 36, *i. e.* (dividing each Side by 12) as 2 to 3. Q. E. D.

Or ſhorter;

The □ of CB is $=2$, by *Schol.* of *Prop.* 49. and the □ of EC $=4$. Therefore the □ of EB $=6$. But the □ of EF is $=9$. Therefore the □ of EB is to the □ of EF as 6 to 9, *i. e.* as 2 to 3. Q. E. D.

CONSECTARY.

THerefore if the Diam. EF be made $=a$, the Side EB will be $=\sqrt{\frac{2}{3}aa}$.

PROPOSITION LVII.

† Eucl. 14. lib. xiii.

THE Side of the Octahedron † is in power one half of the Diameter of the circumſcribed Sphere.

Demonſtration.

For ſince by the Geneſis of the *Octahedron*, Def. 22. (ſee *Fig.* 44. *N.* 2.) CA, CB, CF, *&c.* are ſo many Radii of great Circles, if for the Radius you put a, the Square of AF will be $=2aa$, by the *Pythag. Theor.* But the Square of the Diam. FG $=2a$ is $4aa$. Therefore the Power of the Side is to the Power of the Diam. as 2 to 4, *i. e.* as 1 to 2. Q. E. D.

Shorter ;

Becauſe AF is alſo the ſide of a Square inſcribed in the greateſt Circle, by the Geneſis of the *Octahedron*, the □ of AF will be, by *Prop.* 50. to the □ of FC, as 2 to 1. Therefore to the Square of FG as 2 to 4, by *Prop.* 35. *Q. E. D.*

CONSECTARY.

THerefore if the Diam. of the Sphere be made *a*, the ſide of the *Octahedron* AF will be $\sqrt{\frac{1}{2}aa}$.

PROPOSITION LVIII.

*THE ſide of the Hexahedron or Cube * is in Power ſubtriple or $\frac{1}{3}$ of the Diameter of the circumſcribed Sphere.*

* *Eucl.* 15. *lib.* xiii.

Demonſtration.

Making *a* equal to GF or FE the ſide of the inſcribed Cube (*Fig.*98.) the Square of the Diagonal GE of the Baſe of the Cube will be $2aa$ by the *Pythag. Theor.* and by the ſame reaſon the Square of the Diam. of the Cube and the circumſcribed Sphere GD will be = to the □ of GE + □ of DE = $3aa$. Q. E. D.

CONSECTARIES.

I. THerefore if the Diameter of the Sphere be made = *a*, the ſide of the Cube AB will be $\sqrt{\frac{1}{3}aa}$.

II. The Diameter of the Sphere is equal in power to the ſide of the *Tetrahedron* and Cube taken together. For if the Power of the Diam. of the Sphere be made *aa*, the Power of the ſide of the *Tetrahedron* will be $\frac{2}{3}aa$ by *Conſ. Prop.* 56. and the Power of the ſide of the Cube $\frac{1}{3}aa$, by the preſ. *Conſect.* 1. Wherefore theſe two Powers jointly make *aa*. *Q. E. D.*

PROPOSITION LIX.

*THE ſide of the Dodecahedron * is equal in Power to the greater part of the ſide of the Cube divided in mean and extreme Proportion.*

* *Eucl. Conſect.* 1. *Prop.* 17. *lib.* xiii.

Demonſtration.

For if to the ſide of the Cube AB (*vid. Fig.* 45. *N.* 1.) and to its upper Baſe ABCD you conceive accommodated or fitted a regular Pentagon according to the Geneſis of a *Dodecahedron* laid down in *Def.* 22. and with the Interval B*e* you deſcribe the Arch *ef*, and conceive the ſtraight Line *ef* to be drawn, the Triangles AB*e* and A*ef* will be equi-angular; (for the Angles at A and B being 36°, and A*e*B 108°, *ef* being drawn, the Angles B*ef* and B*fe* are each 72°; therefore A*ef* the remaining Angle will be 36°) wherefore as AB to B*e* (*i. e.* B*f*) ſo A*e* (*i. e.* B*e* or B*f*) to A*f*. Therefore the ſide of the Cube AB is divided in mean and extreme Proportion in *f*, and B*e* the ſide of the *Dodecahedron* is = to the greater part B*f*. *Q. E. D.*

SCHOLIUM.

HEnce we have a new way of cutting a given right Line into mean and extreme Proportion; for if the part (A*e*B) of a regular Pentagon be applied to the given Line by means of the Angles A, B, of 36° each, and with the Diſtance B*e*, the part B*f* be cut off, the thing will be done. Now this Angle of 36° may be had geometrically, if any other regular *Pentagon* be inſcribed in a Circle, and a Chord like to AB being drawn, you make the Angles at A and B each equal to the Angles at the Extremes of that Chord, by *Prop.* 23. *lib.* 1. *Eucl.*

PROPOSITION LX.

* Eucl. Prop. 16. Coroll. lib. xiii.

*THE ſide of an Icoſahedron * is equal in power to the ſide of a Pentagon in a Circle containing only five ſides of the Icoſahedron; and the Semi-diameter of this Circle is in power ſubquintuple or* $\frac{1}{5}$ *of the Diameter of the Sphere circumſcribing the Icoſahedron.*

Demonſtration.

Both theſe are evident from the Geneſis of the *Icoſahedron* in *Def.* 22. The firſt immediately from hence, becauſe all the other ſides of the Triangles (*Fig.* 99.) A*a*, B*a*, *&c.* are made equal to the ſide of the Pentagon AB by *Conſect.* 4. *Prop.* 51. The latter from this Inference; If for OA, the Radius of the Circle, you put *a* (ſince the ſide of the Pentagon, which here is alſo the ſide of the *Icoſahedron*, will be equal in power to the Radius and Side of the Decagon taken together by the aforeſaid *Prop.*) the Altitude OG will be the ſide of the Decagon $= \sqrt{\frac{5}{4}aa} - \frac{1}{2}a$ (by *Conſect.* 1. *Prop.* 54.) to which the equal inferior part *o*H being added, and the intermediate Altitude O*o* $= a$, you will have the whole Diameter of the circumſcribed Sphere GH $= a + 2\sqrt{\frac{5}{4}aa} - \frac{2}{2}a$, i.e. $2\sqrt{\frac{5}{4}aa}$, i.e. $\sqrt{\frac{20}{4}aa}$, i.e. $\sqrt{5aa}$: and ſo the Square of the Diameter of the Sphere will be $5aa$: But the □ of the Radius AO $= aa$. Therefore the Square of the Diameter of the Sphere is to the Square of the Semi-diameter of the Circle, containing the five ſides of the *Icoſahedron*, as 5 to 1. *Q. E. D.*

SCHOLIUM.

THAT a Sphere, deſcribed on the Diameter GH, will paſs thro the other Angles of this *Icoſahedron*, is evident; for aſſuming the Center between O and *o*, the Radius FG will be $= \sqrt{\frac{5}{4}aa}$. But FA is alſo $= \sqrt{\frac{5}{4}aa}$; for the □ of FO is $= \frac{1}{4}aa$, and the □ of AO $= aa$: Therefore the Sum is $= \frac{5}{4}aa =$ □ of FA. *Q. E. D.*

CONSECTARY I.

THerefore, if the Radius of the Circle ABCDE be ſtill *a*, you will have the whole Altitude OG $= \sqrt{\frac{5}{4}aa} - \frac{1}{2}a$, and the ſide of the *Icoſahedron* $= \sqrt{\frac{5}{2}aa - \sqrt{\frac{5}{4}a^4}}$, by *Conſect.* 1. & 2. *Prop.* 54. and the Diameter of the circumſcribed Sphere $2\sqrt{\frac{5}{4}aa}$, as is evident from the Demonſtration.

CONSECTARY II.

Being a general one of the five laſt Propoſitions.

IF AB (*Fig.* 100) be the Diameter of a Sphere * divided in D, ſo that AD be $\frac{1}{3}$ AB; then (having erected the Perpendicular DF) BF will be the ſide of the *Tetrahedron* by *Prop.* 56. and AF the ſide of the *Hexahedron* by *Prop.* 58. *Conſect.* 2. and BE or AE (erecting the Perpendicular CE from the Center) will be the ſide of the *Octahedron* by *Prop.* 57. Now if AF be cut in mean and extreme

* *Eucl. Prop.* 18, *and laſt. lib.* xiii.

treme Proportion in O, you will have AO the ſide of the *Dodecahedron* by *Prop*. 59. Laſtly, if you erect BG the double of CB; HI will be the double of CI, and the □ of HI = 4 □ of CI; conſequently the □ of CH or CB = 5 □ of CI. Therefore the □ of AB (the double of CH) is alſo = to 5 □ of HI (which is the double of CI.) Therefore HI is the Radius of the Circle circumſcribing the Pentagon of the *Icoſahedron*, and IB the ſide of the Decagon inſcribed in the ſame Circle, and HB the ſide of the Pentagon, and alſo the ſide of the *Icoſahedron*, by *Prop*. 60.

BOOK II.

SECT. I.

Containing DEFINITIONS.

DEFINITION I.

IF a Cone ABC (*Fig.* 101.) be conceived to be cut by a Plane at right Angles to any ſide BA thereof; the Plane EGFHE ariſing by this Section, and terminated on the external Surface of the Cone by the curve Line HEG, *&c.* was antiently by *Euclid*, *Archimedes*, &c. called a *Conick Section:* and if the Sides of the Cone AB and BC made a right Angle at B, (*N.* 1.) the Section was particularly called the *Section* of a *right-angled Cone*; but if it made an obtuſe Angle, (*N.* 2.) it was called the *Section of an obtuſe-angled Cone:* if, laſtly, it made an acute one, (*N.* 3.) it was called the *Section of an acute-angled Cone.*

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DEFINITION II.

BUT afterwards their Succeſſors, particularly *Apollonius Pergæus*, from the Properties of theſe Curves, which their Predeceſſors had happily diſcovered, found that all of them might be generated in one and the ſame Cone, whether right angled, obtuſe-angled, or acute-angled, if the Section EF (*Fig.* 102.) be made, in the firſt caſe, parallel to the oppoſite Side BC; in the ſecond caſe, if it meets that Side produced upwards; and in the third, when it meets downwards the Diameter AC of the Baſe produced to D. And gave new Names (for the old ones would do no more now) to theſe Sections, ſo as to diſtinguiſh them one from another, nominating them from their Properties hereafter

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after demonſtrated, calling the firſt a *Parabola*, the ſecond an *Hyperbola*, and the third an *Ellipſis*.

DEFINITION III.

BUT it is evident, that only the Plane making the Section of the ſecond Caſe, being produced according to the Line FED, (*Fig.* 103.) will cut the vertical or oppoſite Cone *a*B*c*, comprehended under the Sides AB, CB, *&c.* continued onwards, and there produce another Section oppoſite to the former; whence theſe Sections, *viz.* GEHG and *gehg*, are called *oppoſite Sections.*

DEFINITION IV.

BEſides theſe Names of the Sections, there are ſeveral others made uſe of to denote various Lines drawn and conſidered both within and without thoſe Sections, the chief whereof we ſhall here explain. And firſt of all, the Line EF ſo let fall thro the middle of the Section from its top E (which is called the *Vertex* of the Section) to the Diameter of the Baſe of the Cone AC (produced if occaſion be) that it ſhall biſect the Line GH and all others parallel to it, is in general called the *Diameter* of that *Section*; and particularly the *Axis* of the *Section*, if it biſects them at right Angles or perpendicularly; like as thoſe Lines GH, KN, *&c.* which are cut in any Angles by the Diameters, but at right Angles by the Axis, are called *Ordinates*, or *ordinate Applicates*, and their halves, FG, IK, *&c. Semi-ordinates*, (or ſome alſo call the latter *Ordinates*, and the former *double Ordinates*) and the Parts EF, EI of the Axis or Diameter, *&c.* are called *Abſciſſæ*, (by ſome, *intercepted* Axes, or Diameters.)

DEFINITION V.

IN the Hyperbola, the Continuation of the Axis or Diameter ED, 'till it meets the oppoſite ſide *c*B, *viz.* to the Vertex of the oppoſite Section, is particularly called the *Latus tranſverſum* of the Hyperbola, to which there anſwers in the Ellipſis the Axis or longeſt Diameter; and ſo by latter Authors is called by the ſame Name, but by *Apollonius* the *tranſverſe Axis*, or the *tranſverſe Diameter*: like as the ſhorteſt Axis or Diameter is called the *conjugate Diameter*, and the middle Point O is called the *Center of the Section*, or of the *oppoſite Sections*.

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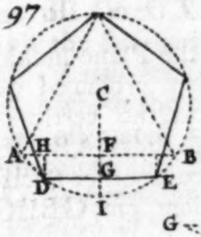

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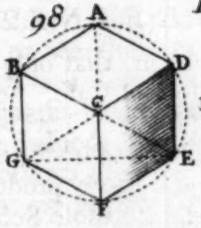

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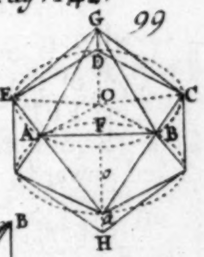

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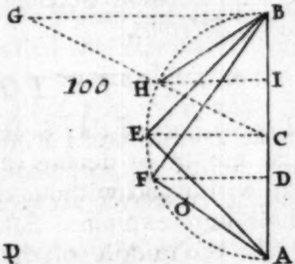

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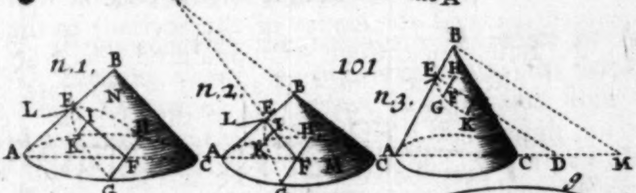

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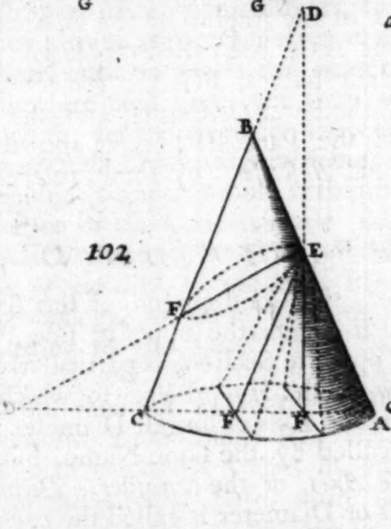

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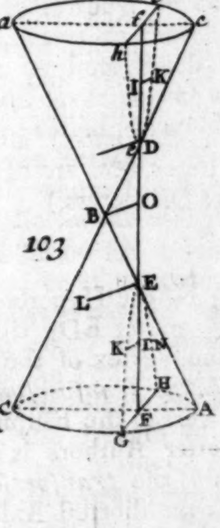

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DEFINITION VI.

THey called alſo a certain Line EL (*Fig.* 101.) by the name of *Latus Rectum*, which is particularly to be found in all the Sections, as we ſhall hereafter ſhew: And becauſe this *Latus Rectum* is a ſort of a Rule or Meaſure, according to which the Squares or Powers of the Ordinates uſed to be eſtimated or valued, (as we will ſhew in its place;) therefore the Antients uſed to call it, by a Periphraſis, *Linea ſecundum quam poſſunt Ordinatim applicatæ*; or, *the Meaſure of the Powers of the Ordinates*: By ſome latter Authors it is called the *Parameter*. Now a mean Proportional PQ found between this *Latus Rectum* and the *Latus Tranſverſum* (*Fig.* 104. *N.* 1, and 2.) [ſee alſo hereafter, *Conſect.* 2. *Prop.* 8.] and drawn through the Center O, parallel to the Ordinates, is called the *ſecond Axis* or *Diameter*; or, *a conjugate Diameter of the Hyperbola.*

p. 142.

146.

DEFINITION VII.

NOW if we conceive the conjugate Diameter or Axis PQ brought down to the Hyperbola, ſo that its middle Point O touches its Vertex in E; and from the Center O you draw the right Lines OR, OS, through the ends *p*, *q*, of this Tangent-Line; theſe are the Lines which *Apollonius*, *Prop.* 1. *lib.* 2. demonſtrates, that if they be continued, they always approach nearer and nearer to the Curve GEH, and come ſo much the nearer by how much the farther they are continued, yet they will never concur with it, or touch it; for which reaſon they are called *Aſymptotes*, or *non-coincident Lines*; and by ſome, the *Intactæ*. Which Non-coincidence appears moſt manifeſtly where the hyperbolick Section of the Cone is made parallel to the triangular Section ABC, through the Axis of the Cone (*N.* 3.) along the Line *ef* parallel to the Axis BF: For if we conceive the Hyperbola *geh* to move forwards always parallel to itſelf, according to the Direction of the equal and parallel Lines *g*G, *f*F, and *h*H, till it ſtands in the Poſition GEH; it is manifeſt that the curve Line GEH is diſtant on both ſides from the right Lines BC, BA, the Length of the verſed Sine of the equal Arches *h*C and *g*A, in the Circumference of the circular Baſe, while in the mean time it is evident that they approach nearer and nearer to it. So that hence there flow the following

CONSECTARIES.

I. IN this Caſe the Sides of the Cone are the Aſymptotes of the Hyperbola, while it is manifeſt, that the Point B is its Centre, and EB half the tranſverſe Diameter; which appears from *N.* 1, and 2. of the preſent *Fig.* for the Section *ef* being made parallel to the Axis of the Cone DF (by *Def.* 5.) *de* (which in the Caſe of *N.* 3. would coincide with *dq*) is the tranſverſe Diameter, but the Triangles *dpq* and O*p*E are equiangular; and conſequently, as *p*E to $\frac{1}{2}$ *pq*, ſo is OE to $\frac{1}{2}$ *dq*.

II. The Lines AG and HC (*N.* 3.) are equal, as being the verſed Sines of equal Arches; and in like manner (*N.* 1. and 2.) RG and HS are equal, ſince FR and FS, as well as E*p* and E*q* are ſo alſo: For,
as OE to $\left.\begin{matrix}Ep\\Eq\end{matrix}\right\}$ ſo is OF to $\left.\begin{matrix}FR\\FS\end{matrix}\right\}$ and the Semi-Ordinates FG and FH are alſo equal.

III. Conſequently the Rectangles of RG into GS, and of HS into HR are equal, *&c.* all which hereafter we will more univerſally demonſtrate.

DEFINITION VIII.

146. * *Archim. de Conoid. & Sphær. Def.* 1.

IF a parabolick Plane * HEGFH (*Fig.* 105. *N.* 1.) together with a Triangle HEG inſcribed in it, and a Rectangle HL circumſcribed about it, be conceived to be moved round about the Axis EF on the Point F; it will be evident, that by the Triangle there will be generated a Cone, by the Rectangle a Cylinder, and by the Parabola a parabolick Solid; which, with the comprehended Cone, and the comprehending Cylinder, will have the common Baſe HIGK, and the ſame Altitude EF, and was by *Archimedes* named a *Parabolick Conoid.*

DEFINITION IX.

† *Archim. de Conoid. & Sphær. Def.* 3.

MOreover, if an † Hyperbolick Plane HEGFH (*N.* 2.) together with the inſcribed Triangle HEG, and another circumſcribed one ROS, made by the Aſymptotes OR, OS, be conceived to be turned about the common Axis OEF on the Point F; it is evident, that by the inſcribed Triangle, a comprehended Cone will be deſcribed; and, by the Hyperbola, an *hyperbolical Conoid,* upon the ſame Baſe HIGK, and of the ſame Height EF; and by the Triangle ROS, another Cone, which *Archimedes* calls the comprehending Cone, whoſe Baſe is RTSV, and its Altitude compoſed of the Axis of the Hyperbola EF, and half the tranſverſe Axis OE (which *Archimedes* called the Additament of the Axis of the Hyperbola) and which we may commodiouſly divide into two Parts, *viz.* into the Cone OPMQL, whoſe Baſe has for its Diameter the conjugate Axis PQ, and its Altitude equal to half the tranſverſe Axis; and into a *Curti-Cone*, or truncated Cone, terminated by the two Baſes PMQL and RTSV, but anſwering in altitude to the Conoid and inſcribed Cone: from which, as comprehending it, if you take away the included Conoid, there will remain the hollow Curti-Cone, terminated below by the *Annulus*, or Ring RGIVHSKT, and above by the circular Baſe PMQL, and generated in the Circumvolution by the intermediate Lines EP, GR, &c. or the mixti-linear Plane EGRP. Now if, inſtead of this comprehending Cone, we ſuppoſe a circumſcribed Cylinder on the ſame Baſe, and of the ſame Altitude with the Conoid and included Cone, you'll have every thing, like as in *Def.* 8.

CONSECTARIES.

NOW if we ſuppoſe the 1ſt Caſe of *Def.* 9. to be expreſſed by the *Fig.* of *Def.* 7. *N.* 3. and conceive the preſent Figure brought thence to be turned round about the Axis BEF, (*Fig*, 106.) we may deduce what follows, in the room of Conſectaries, for the Foundation of our future Demonſtrations.

I. The Lines EQ, RS, HC, &c. of the mixti-linear Space, comprehended between the Hyperbola and the Aſymptotes, (*viz.* the Exceſſes of the Ordinates) although they are unequal; and, as they deſcend, always grow leſs: yet in this Circumvolution they will deſcribe equal circular Spaces, *viz.* EQ, a whole Circle, (or circular Plane) but RS and HC, &c. circular *Annuli*, or Rings, all of the ſame bigneſs; which will thus appear to any one who compares this Figure with the former: Since the Spaces generated by the Lines EQ, FC, &c. are as the Squares of thoſe Lines, and the Square of F*h* or FC, (ſee *Fig.* 104. *N.* 3.) exceeds the Square of *fh*, or FH, by the Square F*f*, or E*e*, or EQ; conſequently the Quantity generated by FC, will exceed that generated by FH, by the Exceſs of that generated by EQ; and alſo that generated by FH by the Exceſs of that generated by HC:

It is manifeſt, that the Circle generated by EQ will be equal to the annular Space generated by HC; and the ſame will, in like manner, be evident of any Spaces produced by RS.

p. 144 II. Therefore the hollow detruncated Cone, generated by the Space EHCQ, according to *Def.* 9. will be equal to a Cylinder generated by the Rectangle FIQE; for all the Indiviſibles of the one are equal to all the Indiviſibles of the other, by *Conſect.* 1.

DEFINITION X.

* *Archim. de Conoid. & Sphær. Def.* 6.

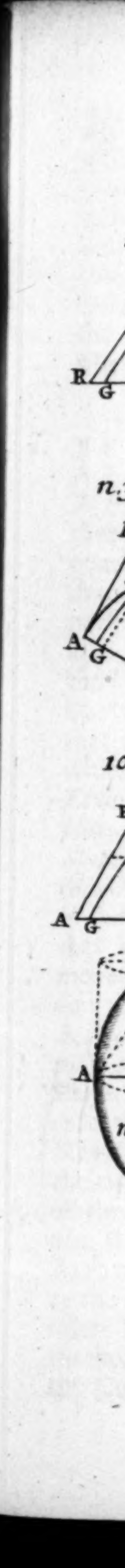
IF, laſtly, * an elliptical Plane be turned about either of the Axes, *viz.* the longeſt DE (*Fig.* 107. *N.* 1.) or the ſhorter AB, (*N.*2.) there will be thence formed an elliptical Solid, called by *Archimedes* a Spheroid; which in the firſt Caſe, will be an oblong, or erect one; in the other, a flat or depreſſed one: And it is ſelf-evident, that if before this Circumvolution of the Ellipſis, there be inſcribed in one of its halves a Triangle, and alſo a Rectangle circumſcribed about it, having the ſame Altitudes and Baſes with the Semi-Ellipſis, there will afterwards in the Circumvolution be deſcribed by the Triangle a Cone, and by the Rectangle, a Cylinder; to which hereafter we will alſo compare the half Spheroid; as alſo both the Conoids, with their inſcribed and circumſcribed Cones and Cylinders.

DEFINITION XI.

p. 150. IF upon the right Line AE (*Fig.* 108.) you conceive a Wheel, or Circle, to roll, until its Point A, with which it touches the ſaid Line, comes to touch it again in E; the Circle will meaſure out the Line AE, equal to its Periphery: but the Point A, by its Motion, will deſcribe the curve Line AFE, which is called a *Trochoid*, or *Cycloid*; and the Area which this Curve, with the Subtenſe AE, comprehends, is named a *Cycloidal Space*; and the Circle, by whoſe Motion they are determined, is called the *generating Circle.*

CONSECTARY.

IT is evident from the Geneſis of this Curve, that the deſcribing Point *a* will always be as much diſtant, according to the Periphery of the Circle, from the Point of Contact *d*, or *c*, as the Point A, in the right Line AE paſſed over, is from the ſame Point of Contact, *i. e.* if the Point *d* be diſtant



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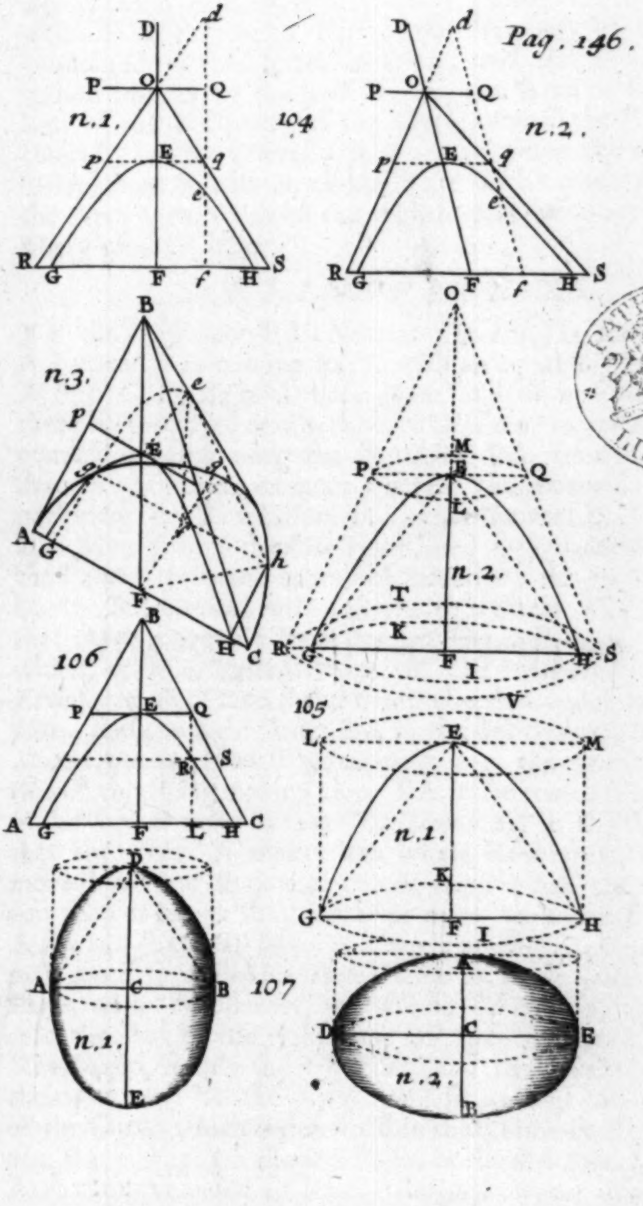

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diſtant from A the fourth Part of the Line AE, the Arch *da* will alſo be the fourth Part of the Periphery of the Circle conſidered in this ſecond Station; and the Point *c* being diſtant from A by the half of AE, the Arch *ca* will be alſo half of the Periphery of the Circle, and ſo the Point *a* will coincide in the Curve with F; and when the Point *e* is diſtant from E only an eighth Part of the whole Line AE, the Arch *ea* will alſo be the eighth part of the whole Periphery of the Circle.

DEFINITION XII.

IF the right Line BA (*Num.* 1. *Fig.* 109.) be moved round one end B remaining fixed, with an equable Motion from A thro' C, D, E, to A back again, and, in the mean while, there be conceived another moveable Point to move with an equable Motion along the ſaid Line BA, from B to A; ſo that in the ſame moment wherein the moveable Point A performs one Revolution, this other moveable Point ſhall alſo have paſſed thro' its right-lined way, coinciding with the Point A returned to its firſt Situation: this Extremity A, by its Revolution, will deſcribe the Circle ACDEA, and that other moveable Point another Curve B, 1, 2, 3, 4, *&c.* which, with *Archimedes*, we will call a *Helix*, or *Spiral Line*; and the Plane Space comprehended under this Spiral Line, and the right Line BA in the firſt Station, is called a *Spiral Space*. Now if we ſuppoſe, *e. g.* the rightlined Motion of the Point moving along BA to be twice ſlower than in the former caſe, ſo that (ſee *Numb.* 2.) in the ſame time that the Point A makes one whole Revolution, the other moveable Point ſhall come to F, moving half the way BA, and then at length ſhall concur or meet with the Extremity A; when that ſhall have performed another Revolution, we may conceive a double ſpiral Line to be deſcribed, whoſe Parts, with *Archimedes*, we will ſo diſtinguiſh, that as he calls the Part of the right Line BF, paſſed over in the firſt Revolution, ſimply the *firſt Line*, and the Circle made by the right Line BF the *firſt Circle*; ſo we will call that Part of the Curve which is deſcribed in that Time or Revolution, *viz.* B 1, 3, 6, 9, 12, the *firſt Helix*, or the *firſt Spiral*, and the Area comprehended under it, the *firſt Spiral Space*: And as the other Part of the right Line FA, paſſed over in the other Revolution, is called the *ſecond Line*, and the Circle marked out by the whole Line BA, the *ſecond Circle*; ſo the Curve deſcribed in the mean while, *viz.* 12, 15, 18, 21,

p. 150.

may be called the *ſecond Spiral Line*, and the Space comprehended under it, the *ſecond Spiral Space*, and ſo on. From theſe Definitions, there flow the following

CONSECTARIES.

I. IT is evident, that the Lines B 12, B 11, B 10, *&c.* drawn to the firſt or ſecond Spiral, making equal Angles, and * B 12, B 10, B 8, *&c.* or B 12, B 9, B 6, *&c.* are arithmetically proportional.

* *Archim. Prop.* 2. *of Spirals.*

II. Lines drawn any how to the firſt Spiral, as B 7, B 10, *&c.* are to one another, as the Arches of the Circles intercepted between BA and the ſaid Lines † B 7, B 10, *&c.* which is alſo evident to any one who conſiders what we have ſuppoſed in the *Def.* for in the ſame time as the End A paſſes over ſeven Parts of the Circle, the other moveable Point will alſo run over ſeven Parts of the right Line BA, *&c.*

† *Archim. Prop.* 14.

III. Laſtly, Right Lines drawn from the initial * Point B, to the ſecond Spiral, *e. g.* B 19 and B 22 (*Num.* 2.) will be one to another as the aforeſaid Arches, together with the whole Periphery added to both Sides: for at the ſame time the Extremity A runs thro' the whole Circle, or 12 Parts, and moreover 7 Parts (*i. e.* in all 19 Parts;) in the ſame time the other moveable Point paſſes thro' 12 Parts of the right Line BA (in this caſe divided into 24 Parts) and moreover 7 Parts, that is, in all 19; and ſo in the others.

* *Archim. Prop.* 15.

DEFINITION XIII.

IF an indefinite right Line (or Ruler) BAF (*Fig* 110.) moves ſo within the right Angles ADC, CDE, form'd by the Legs of a double Square, that it always paſſes thro' [or touches] a Point C (fixed in the Leg KD) of the Square, and the Point A fixed in it always ſlides along the Leg DA of the Square: By this motion, the Extremity B of the moveable Line (or Ruler) will deſcribe the Curve B, *b*, *b*, *&c.* called by its Inventor *Nicomedes*, a *Chochoid:* Being attended with this Property, that the right Lines CB, C*b*, drawn from the Center C to the Curve, are not equal, as in the Circle; but yet have all their Parts AB, *ab*, intercepted between the Curve and the Directrix, or Side of the Square AE, equal, as is evident from the Generation of the Curve.

DEFINITION XIV.

IF in a Circle, (*Fig.* 111. *N*1.) whoſe Diameters cutting one another at right Angles, are AB, CD, you take equal Arches BE or B*e*, and BF or B*f*, *&c.* and from E or *e* you draw the Perpendiculars EG or *eg*, and through theſe from D to F or *f*, *&c.* you draw the tranſverſe Lines DF or D*f*; the ſeveral Points of Interſection H, *h*, *&c.* decently connected together, will exhibit the Curve-Line DBH (which may be continued alſo without the Circle, if you pleaſe) and which is commonly attributed to *Diocles*, and called a *Ciſſoid*.

SCHOLIUM.

IF any one requires an organical way of deſcribing this Curve, he may do it by duly conſidering the other Figure (*N.* 2.) For it is plain, that if the two croſs Rulers CE, DF, moveable about C and D, do firſt cut one another in B, and the Side GE of the Square be placed ſo as to coincide with LB, and ſo the three Points O, B, E; and afterwards the Point O of Interſection moves from B to L, and at the ſame time the Ruler CE moves the Side GE of the Square along the Periphery BD, always parallel to itſelf, that the continual Interſection H of this Side of the Square, and the other croſs Ruler DF will deſcribe the required Curve BHD, *&c.*

DEFINITION XV.

THE right Line AE being divided (*Fig.* 112.) into the equal Parts AB, BC, *&c.* if from the Points of the Diviſion A, B, C, *&c.* you draw the parallel Lines A*a*, B*b*, C*c*, *&c.* in geometrical Progreſſion, (as, *e. g.* A*a*, 1, B*b*, 4, C*c*, 16, D*d*, 64, *&c.* or B*b*, 10, C*c*, 100, D*d*, 1000, E*e*, 10000, *&c.*) and further biſecting AB, BC, *&c.* in 𝔉, 𝔊, ℌ, ℑ, you let fall mean Proportionals between the next Collaterals 𝔉*f*, 𝔊*g*, ℌ*h*, ℑ*i*, *&c.* and continue to do ſo till the Parallels are brought very near to one another; the curve Line drawn thro' the Extremities of theſe Lines *af*, *lg*, *ch*, &c. will be the *Logarithmical Line* of the Moderns, whoſe Properties and Uſes are very excellent.

SCHOLIUM.

AMong theſe Uſes, that is none of the leaſt, from which this Curve borrows its Name, *viz.* in ſhewing the Nature and Invention of Logarithms. For, *e. g.* 1. If this Line was accurately delineated in a large Space, the Parts AB, BC, *&c.* being taken ſo large, that they might be ſubdivided, not only into 100 or 1000, but into 10000 or 100000, *&c.* Parts; if AB be made 100000 (and ſo A 00000) AC would be 200000, AD 300000, *&c.* In the mean while, there anſwer to theſe as primary Logarithms in arithmetical Progreſſion, the geometrical proportional Numbers A*a* 1, B*b* 10, C*c* 100, D*d* 1000, E*e* 10000, *&c.* Whence, 2*dly*, Any intermediate Number being given, we may find its Logarithm, as ſuppoſe the Number 982; for having cut off this Number from D*d* by a geometrical Scale, which let be the Line DM, if you draw M*n* parallel to AD, and *n*N Parallel to DM, then will AN, on the ſame Scale, be the Logarithm ſought, and contrariwiſe. But if, 3*dly*, It ſeems difficult to delineate a Figure ſo large, yet, at leaſt, the clear Conception of ſuch a Delineation evidently ſhews the arithmetical Method, which thoſe ingenious Men have made uſe of, who have made the Tables of Logarithms with a great expence of Labour and Pains, *viz.* by finding continual mean Proportionals, *i.e.* arithmetical ones, between any two Logarithms already known, and geometrical ones between two vulgar Numbers anſwering to them, *&c.* by comparing what we have noted in *Schol.* 2. *Prop.* 20. *Lib.* 1. And we will note, 4*thly*, out of *Pardies*, that ſince the Logarithms of Numbers, diſtant from one another by a decuple Proportion, differ by the Number 100000; having found the Logarithms of all the Numbers from 1000 to 10000, you will at the ſame time have all the Logarithms of all the other Numbers that are between 100 and 1000, between 10 and 100, and between 1 and 10, only changing the Characteriſtick, and leſſening it in the firſt caſe by Unity; in the ſecond, by 2; in the third, by 3; as, *e.g.* if you had found 399563 the Logarithm of the Number 9900, the Logarithm of the ſubdecuple Number 990 would be (*viz.* ſubſtracting from the former 100000) 299563, and the Logarithm again of the Subdecuple of this 99, would be 199563, *&c.* Thus in the Chiliads of *Briggs* to the Number

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990	——	2, 99563, 51946
99	laſtly——	1, 99563, 51946

But there will not ariſe ſuch advantage for making Logarithms by this Obſervation, as it may at firſt ſight ſeem to pro-

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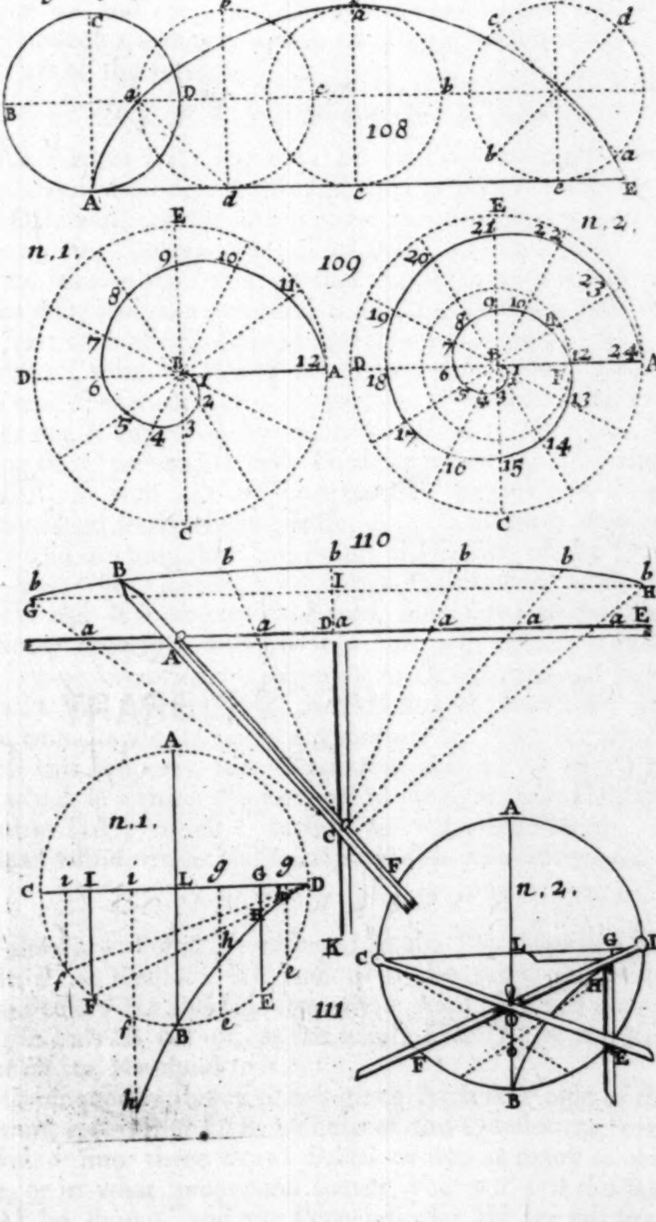

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promiſe, becauſe there are 9000 Numbers between 1000 and 10000, whoſe Logarithms muſt be found alſo; and but 900 between 100 and 1000, and but 90 between 10 and 100, and but 9 between 1 and 10; and ſo in all 999, which is not the ninth Part of the former.

DEFINITION XVI.

p. 158.

IF the Radius AD (*Fig.* 113.) be conceived to move equably about the Point A through the Periphery of the Quadrant DB, while in the mean time the Side of the Square DC remaining always parallel to itſelf, deſcends alſo with an equal Motion thro' DA, ſo that in the ſame moment the Radius AD, and the aforeſaid Side DC, ſhall fall upon the Baſe AB; then the continual Interſection of AD and DC will deſcribe a Curve, or if any one ſhould think, that, according to this Generation, the Proportion of a right Line to a circular one is ſuppoſed by a ſort of *Petitio Principii*, or begging the Queſtion, it may be deſcribed thus: The right Line DA, as well as the Quadrant DB, being divided into as many equal Parts as you pleaſe, (*e.g.* here both of them into 8) and drawing thro' the Points of Diviſion of the Quadrant from the Center A ſo many Radii, and thro' the Points of the Radius parallel Lines, the Points of Interſection being orderly connected together, will exhibit a curve Line, whoſe Invention is attributed to *Dinoſtratus* and *Nicomedes* in the fourth Book of *Pappus Alexandrinus*, and which, from its uſe, is called a *Quadratrix*, it having among the reſt this Property, *viz.* That from AB it cuts off a Part AE, which is a third Proportional to the Quadrant DB, and its Radius DA; which hereafter we will demonſtrate. In the mean while, from this Deſcription of it, you have theſe

CONSECTARIES.

I. IF thro' any Point H, aſſumed in the *Quadratrix*, you draw the Radius AHI, and from the ſame Point the Perpendiculars H*b* and H*e*, the whole Arch DB will always be to the Part IB cut off, as the whole Line DA to the Part *b*A cut off, or H*e* equal to it.

II. Conſequently thereſore any given Arch or Angle of the Quadrant, *e.g.* IB or IAB, by help of the Quadratrix, may be divided into three equal Parts, or into as many as you pleaſe, or in what proportion ſoever you will; if the Radius AI be drawn, and the Perpendicular H*e*, let fall from the Point of the Quadratrix H, be divided into three or as many equal Parts as you pleaſe, or in any proportion whatſoever, and thro' the Points of Diviſion, Radii be drawn, for they will divide the Arch as requir'd.

BOOK II.

SECT. II.

CHAP. I.

Of the chief Properties of the Conick Sections.

PROPOSITION I.

* 1. *Property of the Parab. Apoll. Prop.* 11. *lib.* I.

IN *the Parabola* (GKEH, *Fig.* 114.) *the* * *Square of the Semi-ordinate* (IK) *is equal to the Rectangle* IL, *made by the Latus Rectum* EL, *and the Abſciſſa* EI.

Demonſtration.

MAKE the Sides of the Cone that is ſuppoſed to be cut, *viz.* $AB=a$, $BC=b$, and moreover $EB=ra$, and $EI=sb$, and $AC=c$; therefore NI will be $=sc$, by reaſon of the Similitude of the Triangles BCA and EIN; and EP or $IO=rc$, by reaſon of the Similitude of the Triangles ABC and EBP. Therefore the ▭ NIO $=rscc=$ □ of IK, by the *Schol.* of *Prop.* 34. (*N.* 3.) and *Prop.* 17. *Lib.* 1. Now if a Line be ſought, which with the Abſciſſa EI ſhall make the ▭ IL $=$ □ of IK, you will have it by dividing the ſaid Square by the Abſciſſa EI, *viz.* $\frac{rscc}{sb}$, i. e. $\frac{rcc}{b}=EL$. And this is called the *Latus Rectum*, *viz.* in relation to the Abſciſſa EI, with which it makes that Rectangle, which, it's evident, is = □ of IK; and from this Equality the Section has the Name of Parabola, by *Apollonius.*

CONSECTARIES.

I. THIS *Latus Rectum*, expreſſed by the Quantity $\frac{rcc}{b}$, may be found out after a ſhorter way, by making as *b* to *c* (the Side of the Cone parallel to the Section BC, to AC the Diameter of the Baſe) ſo is *rc* (*viz.* the Side EP called by ſome the *Latus Primarium*) to a fourth.

II. But if any one, with *Apollonius*, had rather expreſs this by meer *Data* in the cut Cone itſelf (becauſe *rc*, or the *Latus Primarium* EP, is not a Line belonging to the Cone itſelf) he may eaſily perceive, if the Quantity of the *Latus Rectum* found above be multiplied by the other Side of the Cone *a*, there will be produc'd the equivalent Expreſſion $\frac{racc}{ab}$, which, inſtead of the Proportion above, will furniſh us with this other,

$$\left.\begin{matrix}\text{as } ab\\ AB\times BC\end{matrix}\right\}\begin{matrix}\text{to } cc\\ \square\text{ of } AC\end{matrix}\begin{matrix}::\\ ::\end{matrix}\left.\begin{matrix}\text{ſo } ra\\ EB\end{matrix}\right\}\text{ to a fourth;}$$

which is the very Proportion of *Apollonius* in *Prop.* 9. *Lib.* 1. and confirms our former.

SCHOLIUM I.

HENCE you have an eaſy and plain way of deſcribing a Parabola, having the top of the Axis and the *Latus Rectum* given, *viz.* by drawing ſeveral Semi-ordinates, whoſe extreme Points connected together, will exhibit the Curvity of the Parabola. But you may find as many Semi-ordinates as you pleaſe, if having cut off as many Parts of the Axis as you pleaſe, you find as many mean Proportionals between the *Latus Rectum* and each of thoſe Parts or Abſciſſæ. See *N.* 2, and 3. *Fig.* 47. *Introduct. to Specious Analyſis.*

SCHOLIUM. II.

HENCE alſo we have a new way of deſcribing a Parabola from M. *De Witte*, which is thus: If the right-lined Angle HBG, moveable about the fixed Point B (*Fig.* 115.) be ſo moved, that one of its Legs BH carries with it one End H of the Ruler HG along the fixed Ruler EF, in ſuch manner, that HG always moves from its firſt Situation

DB parallel to itſelf; then the Interſection of the other Leg BG, and the ſaid Ruler HG, continually moving from B towards G, will deſcribe a Curve, which will be the Parabola of the Antients. For, 1. If the Angle HBG (*N.* 1.) be a right Angle, and you make BD or HI $=a$, BI or KG $=b$ (in that ſtation of the Angle and Ruler HG which the Point of Interſection G denotes) then becauſe of the right Angle B, the Line BI $=b$, is a mean Proportional between HI $=a$, and IG or BK; and therefore IG or BK, as an Abſciſſa, is $=\frac{bb}{a}$. Whence, if BK $=\frac{bb}{a}$ be multiplied by BD $=a$, the Rectangle DBK will be $=bb=\square$ of KG, which is the firſt Property of the Parabola we have proved: So that ſince the ſame Inference may be made of any other Point in this Curve, it follows, that this Curve will be the Parabola, BD or HI its *Latus Rectum*, KG a Semi-ordinate, and BK its Axis, *&c.* 2. If the Angle HBG be an oblique one (*Num.* 2.) it may be eaſily ſhewn from what we have ſuppoſed, that the Triangles DBH and BKG will be Equiangular: Therefore as BD (a) to DH, ſc. BI (b); ſo is KG ſc. BI (b) to BK $=\frac{bb}{a}$. Therefore again the ▭ DBK $= bb = \square$ of KG. Q. E. D.

Conſect. III. It is alſo evident in this ſecond caſe, that BK drawn parallel to the Axis, but not thro' the middle of the Parabola, will be a Diameter which will have B for its Vertex, BD for its *Latus Rectum*, and GK for a Semi-ordinate, *&c.*

Conſect. IV. Therefore you may find the *Latus Rectum* in a given Parabola geometrically, if you draw any Semi-ordinate whatſoever IK (*Fig.* 116.) and make the Abſciſſa EF equal to it, and from F draw a Parallel to the Semi-ordinate IK, and from E draw the right Line EK thro' K cutting off FH the *Latus Rectum* ſought; ſince as EI to IK, ſo is EF (*i.e.* IK) to FH, by *Prop.* 34. *lib.* 1. wherefore having the Abſciſſa and Semi-ordinate given arithmetically, the *Latus Rectum* will be a third Proportional.

Conſect. V. Since therefore the *Latus Rectum* found above is $\frac{rcc}{b}$, if you conceive it to be applied to the Parabola ſo as to be the Ordinate LM, then ſhall N be that Point which is called the *Focus*, and LN will be $\frac{rcc}{2b}$, and its Square will be $\frac{rrc^4}{4bb}$; and this divided by the *Latus Rectum* $\frac{rcc}{b}$ will give $\frac{rcc}{4b}$ for the Abſciſſa EN: So that the *Diſtance of the Focus from the Vertex* will be $\frac{1}{4}$ of the *Latus Rectum.*

Conſect. VI. Since therefore EN is $=\frac{rcc}{4b}$, if for EF you put ib, NF will be $=\frac{ib-rcc}{4b}$, whoſe Square will be found to be $i^2b^2-\frac{ricc}{2}+\frac{r^2c^4}{16b^2}$, to which if there be added □ of GF $=ricc$, by *Prop.* 1. the Square of NG will be $= i^2b^2+\frac{ricc}{2}+\frac{r^2c^4}{16b^2}$, whoſe Root (as the Extraction of it, or the Analogy of the Square NF with the Square NG, manifeſtly ſhews) will be $ib+\frac{rcc}{4b}$; ſo that a right Line drawn from the *Focus* to the end of the Ordinate, will always be equal to the Abſciſſa EF + EN, *i. e.* (if EO be made equal to EN) to the compounded Line FO.

SCHOLIUM III.

HENCE we have an eaſier Way of deſcribing a Parabola *in plano*, having the *Focus* and *Vertex* given, *viz.* (*Fig.* 117.) the Axis being prolonged along the Vertex E to O, ſo that EO be = EN; and then move the Square HI from OF to HI, along the Ruler PQ, fixed at right Angles to the Axis at the Point O, ſo that IG, a part of the Thread IGN being equal in Length to the Side IH of the Square, and faſten'd at one end to the Focus N, and at the other to the end I of the Square, may be kept cloſe to the Side of the Square, by means of the Pin G, (which perhaps might alſo be done with the Compaſſes, by a Contrivance which we will hereafter accommodate to the Hyperbola) and the Pin G will deſcribe the Portion RGE of a Curve, which will be a Semi-Parabola; becauſe, by the foregoing Conſectary, as the whole Thread is always = to HI the Side of the Square, ſo the part GN is always neceſſarily equal to the part GH, that is, to the Line FO. Moreover, from the ſame fixth Conſectary, and *Fig.* 116. there flows another eaſy Way of deſcribing a Parabola *in plano* thro' innumerable Points G, to be ſound after the ſame manner, by having the Focus and Vertex given: *viz.* if from any Point F taken in the Axis, you draw the Perpendicular FG, and at the Diſtance FO, you make an Interſection G from the Focus N, which innumerable Points G will be found with the ſame facility, having aſſumed the Axis and *Latus Rectum*, by virtue of the preſent Propoſition: For if the Point F being taken at pleaſure in the Axis, you find a mean Proportional between the *Latus Rectum* and the Abciſſa EF, a Semi-ordinate FG made equal to it, will denote or mark the Point G in the Parabola ſought.

PROPOSITION II.

p. 158.

* I. *Property of the Hyperb. Apoll. Prop. 12. Lib. 1.*

IN the Hyperbola (GKEH, Fig. 118.) * *the Square of the Semi-ordinate* (IK) *is equal to the Rectangle* (IL) *made of the* Latus Rectum (EL) *and the Abſciſſa* (EI) *together with another Rectangle* LS *of the ſaid Abſciſſa* (EI *or* LR) *and* RS *a fourth Proportional to* DE *the* Latus Tranſverſum, (EL) *the* Latus Rectum, *and* EI *the Abſciſſa.*

Demonſtration.

Suppoſe the Side of the Cone AB here alſo $=a$, and BM parallel to the Section $=b$, and the intercepted Line AM $=c$, and EI $=sb$, all according to the Analogy we have obſerved in the Parabola; and NI will, as there, be $=sc$. Making, moreover, MC $=d$, and the *Latus Tranſverſum* DE $=rb$, ſo that DI be $=rb+sb$; then (by reaſon of the Similitude of the Triangles BMC, DEP, and DIO) will EP be $=rd$, and IO $=rd+sd$; and ſo QO $=sd$. Therefore ▭ NIO will be $=rscd+sscd=$ □ of IK; but this Square, divided by the Abſciſſa EI $=sb$, gives $\frac{rscd+sscd}{sb}$ or $\frac{rcd+scd}{b}$; for the Line IS, which, with the Abſciſſa, would make the Rectangle ES = to the ſaid Square of IK. Now therefore, if here alſo we call a Line the *Latus Rec-*

tum, found after the ſame Way as in the Parabola, *viz.* by making

as b .... to c .... ſo is rd to a fourth $\frac{rdc}{b}$

as a Line parallel to the Section .... to the intercepted Diam. ſo is the *Latus Primarium*, to a 4th; but that the other part $\frac{scd}{b}$ will be a fourth Proportional to bc and sd, or to sb, sc, and sd, or (to ſpeak with *Apollonius* as we have done in the Prop.) to rb, $\frac{rcd}{b}$ and sb (for in theſe three caſes you'll have the ſame fourth Quantity $\frac{scd}{b}$.) Wherefore now it is evident, that the Square of the Semi-Ordinate $rscd + sscd$ is equal to the Rectangle IL (made by the *Latus Rectum* $\frac{rcd}{b}$ into the Abſciſſa $sb = rscd$) together with the ▭ LS of this fourth Proportional $\frac{scd}{b}$ into the ſame Abſciſſa sb, which is $= sscd$: Which was to be found and demonſtrated.

CONSECTARIES.

I. HENCE you have in the firſt place the reaſon why *Apollonius* called this Section an Hyperbola, *viz.* becauſe the Square of the Ordinate IK exceeds, or is greater than the Rectangle of the *Latus Rectum* and the Abſciſſa.

II. Since therefore the *Latus Rectum* here alſo, as well as in the Parabola, is found by making as b to c, ſo is rd to $\frac{rdc}{b}$ (*i.e.* as the Parallel to the Section BM is to the intercepted Diam. AM, ſo is the *Latus Primarium* EP to a fourth EL.) If any one had rather expreſs this *Latus Rectum* after *Apollonius*'s Way, he will eaſily perceive, this Quantity being found, and multiplied, both Numerator and Denominator by b the Parallel to the Section, that there will come out the equivalent Quantity $\frac{rbcd}{bb}$, which gives us, inſtead of the former Proportion, this other,

as bb .... to cd .... ſo is rb .... } to a fourth,
□ of BM .... ▭ AMC .... *Latus Tranſverſum* }

which is that of *Apollonius* in *Prop.* 12. *lib.*1. and conſequently herein confirms our former.

III. You may alſo have this *Latus Rectum* geometrically, by finding a third Proportional (as we have done in the Parabola, *Conſect.* 4. *Prop.*1.) to the Abſciſſa EI (*Fig.* 119.) and the Semi-ordinate IK (= EF;) and then a fourth Proportional EL to DI (the Sum of the *Latus Tranſverſum* and Abſciſſa) and FH already found, or IS equal to it, and DE (the *Latus Tranſverſum*) and that will be the *Latus Rectum* ſought.

SCHOLIUM.

FROM this third Conſectary, by having the *Latus Rectum* and Tranſverſe given, we may on the contrary find out and apply as many Semi-ordinates to the Axis as we pleaſe, and ſo deſcribe the Hyperbola thro' (their Ends or) infinite Points: *viz.* If aſſuming any part of the Abſciſſa EI, you make as DE to EL, ſo DI to IS; and then find a mean Proportional IK between IS and the Abſciſſa EI, and that will be the Semi-ordinate ſought: And both this Praxis and the Conſect. may be abundantly proved by ſetting it down in, and making uſe of, the literal *Calculus.*

PROPOSITION III.

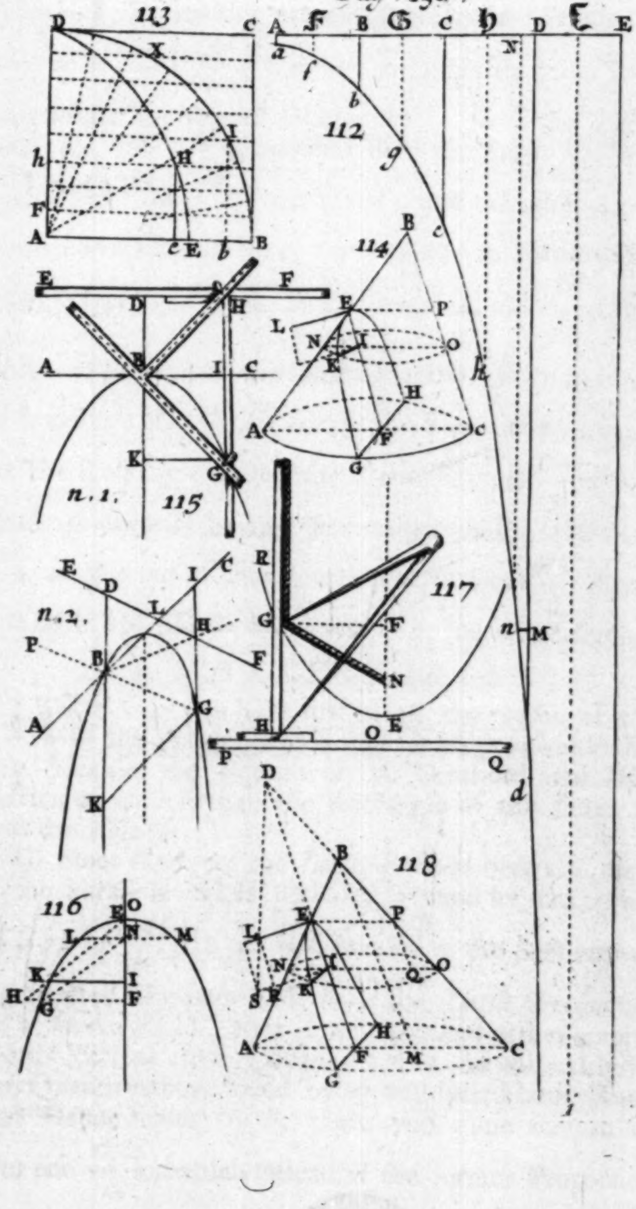
* 1ſt Property of Ellipſe of Apol. Prop.13. lib. 1.

IN the Ellipſis KDEK (*Fig.* 120.) * *the Square of the Semi-ordinate* IK *is equal to the Rectangle* IL *of the Latus Rectum* EL, *and the Abſciſſa* EI *minus another Rectangle* LS, *of the ſame Abſciſſa* EI *or* LR, *and* RS *a fourth Proportional to* DE *the Latus Tranſverſum*, EL *the Latus Rectum, and* EI *the Abſciſſa.*

Demonſtration.

Let the Side of the Cone AB be here alſo $= a$, and BM parallel to the Section $= b$, and the intercepted part AM $= c$, and EI $= sb$; and NI will be again $= sc$, all as in the Hyperbola. And making alſo here, as in the Hyperbola MC $= d$, and the *Latus Tranſverſum* DE $= rb$, ſo that DI may be $rb - sb$; then (by reaſon of the ſimilitude of the Triangles BMC, DEP, and DIO) EP will be $= rd$, and IO $= rd - sd$. Therefore the Rectangle NIO will be $= rscd - sscd = \Box$ of IK. But this Square divided by the Abſciſſa EI $= sb$, gives $\frac{rscd - sscd}{sb}$ or $\frac{rcd - scd}{b}$, for that Line IS,

which



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which with the Abſciſſa would make the Rectangle ES $=$ to the ſaid Square of IK. Now therefore if we call the *Latus Rectum* a right Line, found after the ſame way as in the Parabola, by making according to *Conſ.* 1. *Prop.* 1. as b to c, p. 54 ſo r to a fourth $\frac{rcd}{b}$; i. e. as the Line parallel to the Section ... to the intercepted Diameter, ... ſo is the *Latus Primarium*, &c. It is manifeſt that the *Latus Rectum* is one part of the Line juſt now found; and the other part $\frac{scd}{b}$ is a fourth Proportional to b, c and sd, or (to ſpeak with *Apollonius*, as we have done in the Prop.) to rb, $\frac{rcd}{b}$ and sb, (for there will come out the ſame Quantity $\frac{scd}{b}$;) wherefore now it is evident that the Square of the Semi-ordinate IK is equal to the Rectangle IL (of the *Latus Rectum* $\frac{rcd}{b}$ and the Abſciſſa $sb = rscd$) having firſt taken thence the Rectangle LS, or $sscd$ out of that fourth Proportional $\frac{scd}{b}$ drawn into the ſame Abſciſſa sb; which was to be found and demonſtrated.

CONSECTARIES.

I. HENCE you have firſt of all the reaſon of the name of the *Ellipſis*, which *Apollonius* gave to this Section; *viz.* becauſe the Square of the Semi-ordinate IK is defective of, or leſs than the Rectangle of the *Latus Rectum* and the *Abſciſſa*.

II. Since therefore the *Latus Rectum* here alſo, as well as in the Parabola and Hyperbola, is found by making as b to c, ſo is rd to $\frac{rcd}{b}$ (*i.e.* as BM parallel to the Section, is to the intercepted Diameter AM, ſo is the *Latus Primarium* EP to a fourth EL.) Now if any one had rather expreſs this *Latus Rectum* after *Apollonius*'s way, he will eaſily ſee that the Quantity above found being multiplied both Numerator and Denominator by b, there will come out an equivalent one $\frac{rbcd}{bb}$, which inſtead of the former Proportion, will give this other,

as bb . . . . to cd . . . . ſo rb } to a fourth.
□ of BM . . . ▭ AMC . . . *Latus Tranſverſ.* }

which is the ſame with that we have alſo found in the Hyperbola, and which alſo *Apollonius* has, *Prop.* 13. *lib.* 1.

p. 172.

III. This *Latus Rectum* may alſo be had geometrically, if, 1. In the Hyperbola you find a third Proportional FH to the Abſciſſa EI (*Fig.* 121.) and Semi-ordinate IK (= EF.) 2. And EL a fourth Proportional to DI (the Difference of the *Latus Tranſverſum* and the Abſciſſa) and FH or IS equal to it, and the *Latus Tranſverſum* DE will be the the *Latus Rectum* ſought.

SCHOLIUM.

FROM this third Conſect. having the *Latus Rectum* and *Tranſverſum* given, we may on the contrary apply as many Semi-ordinates to the Axis as you pleaſe, and ſo draw the Ellipſis thro' as many Points given as you pleaſe; *viz.* if, taking any Abſciſſa EI, you make as DE to EL, ſo DI to a fourth IS; then between this IS and the Abſciſſa EI find a mean Proportional IK, and that will be the Semi-ordinate ſought. And this Praxis alſo, and the third Conſect. may be abundantly proved by making uſe of a literal *Calculus*: For *c.g.* here a fourth Proportional to rb, $\frac{rcd}{b}$ and $rb - sb$, will be $\frac{rcd - scd}{b}$; and a mean Proportional between this fourth and sb, will be $\sqrt{rscd - sscd}$, &c.

PROPOSITION IV.

* 2d Property of the Parab. 20 Prop. Apol. lib. 1. Con.

*IN a Parabola * the Squares of the Ordinates are to one another as the Abſciſſæ.*

Demonſtration.

p. 172.

For if EF (*Fig.* 122.) be called ib, as above EI was called sb, ſince the *Latus Rectum* is $\frac{rcc}{b}$; the Square of FG will be $= ricc$. Therefore

□ of IK will be to □ of FG } as s to i or
$rscc$ . . . . $ricc$ } sb to ib. Q.E.D.

CONSECTARY.

HENCE having drawn LO parallel to the Axis or Diameter EF, if it be cut by the tranſverſe Line EG in

M,

M, and by the Curve of the Parabola in K; then will OL, ML, and KL be continual Proportionals. For EF is to EN, as FG to NM or IK, by reaſon of the Similitude of the Triangles EFG and ENM. But the Squares of FG and IK, are in the duplicate Proportion of EF to EN, by *Prop.*25. *lib.*1. and are alſo in the ſame proportion as the Abſciſſæ EF and EI by the preſ. Therefore EF to EI is alſo in the duplicate proportion of EF to EN; *i.e.*

$$\begin{array}{c}EF\\OL\end{array}\text{ is to }\left.\begin{array}{c}EN\\ML\end{array}\right\}\text{ as }\left\{\begin{array}{c}EN\\ML\end{array}\right.\text{ to }\left.\begin{array}{c}EI\\KL\end{array}\right\}\text{ Q. E. D.}$$

PROPOSITION V.

*IN the Hyperbola and Ellipſis * the Squares of the Ordinates are as the Rectangles contained under the Lines which are intercepted between them, and the Vertexes of the Latus Tranſverſum.*

* *Property of the Hyperbola and Ellipſ. Apol.* 21. *lib.* 1.

Demonſtration.

For, if EF (*Fig.* 118, and 120.) be called *ib*, as EI was above called *sb*; then, according to *Prop.*2. and the 3*d* Deduction, will

$$\square\, GF = ricd + sicd \text{ in the Hyperb.}$$
$$ricd - sicd \text{ in the Ellipſ.}$$
$$\text{and the } \sqsubset\!\sqsupset\, DFE = ribb + iibb \text{ in the Hyperb.}$$
$$ribb - iibb \text{ in the Ellipſ.}$$

Therefore the □ of KI is to the Square of GF, as $rscd \pm sscd$, to $ricd \pm iicd$, i.e. as $rs \pm ee$ to $ri \pm ii$. And ▭ DIE is to the ▭ DFE, as $rsbb \pm ssbb$ to $ribb \pm iibb$, i.e. in like manner as $rs \pm ss$ to $ri \pm ii$. Q. E. D.

CONSECTARY I.

IN the Ellipſis this may be more commodiouſly expreſſed apart thus: The Squares of the Ordinates KI and GF are as the Rectangles contained under the Segments of the Diameter (*viz.* DIE and DFE) in which ſenſe this Property is alſo common to the Circle, wherein the Squares of the Ordinates are always equal to the Rectangles of the Segments.

CONSECTARY II.

THerefore, if the *Latus Rectum* be conceived to be applied in the Hyperbola, ſo that N ſhall be the *Focus*; (ſee *Fig.* 123.) then will $LN = \frac{rcd}{2b}$, and its Square will be

$\frac{rrccdd}{4bb}$. But as the □ of KI to the □ of LN, ſo is the ▭ DIE to the ▭ DNE; *i.e.* as $rscd + sscd$ to $\frac{rrccdd}{4bb}$, ſo is $rsbb + ssbb$ to $\frac{rrcd}{4}$. But now the ▭ of the Whole DE, and the Part added EN into the Part added EN; *i.e.* ▭ DNE $(=\frac{rrcd}{4})$ together with the Square of the half CE $(=\frac{rrbb}{4})$ is = □ compounded of the Half and the Part added CN $=\frac{rrcd+rrbb}{4}$, by *Prop.* 9. *lib.* 1. Wherefore CN, the Diſtance of the *Focus* from the Centre, is $=\frac{\sqrt{rrcd+rrbb}}{4}$. But $\frac{rrcd}{4}$ is the fourth part of the ▭ of the *Latus Tranſverſum rb*, and the *Latus Rectum* $\frac{rcd}{b}$ (or the fourth part of the Figure, as *Apollonius* calls it) and $\frac{rrbb}{4}$ is the □ of $\frac{rb}{2}$, *i.e.* of half the *Latus Tranſverſum*. Wherefore we have found the following Rule of determining the *Focus* in an Hyperbola: viz. *If a fourth part of the Figure* (or the Rectangle of the *Latus Rectum* into the *Tranſverſum*) *be added to the Square of half the* Latus Tranſverſum, *and from the Sum you extract the Square Root; that will be the Diſtance* CN *of the Focus from the Center: And hence ſubſtracting half the* Latus Tranſverſum CE, *you will have the Diſtance* EN *of the Focus from the Vertex.*

CONSECTARY III.

IN like manner, in the Ellipſis having applied the Ordinate LM (*Fig.* 124.) that ſo the *Focus* may be in N, the □ of LN will be $\frac{rrccdd}{4bb}$ as above; and by a like Inference ▭ DNE $=\frac{rrcd}{4}$. But now ▭ DNE, together with the Square of the Difference CN, is equal to the □ of the half CE by *Prop.* 8. *lib.* 1. and conſequently the □ of CN is = □ of CE — ▭ DNE, that is, $\frac{rrbb}{4} - \frac{rrcd}{4}$. Wherefore CN, the diſtance of the *Focus* from the Centre, is = $\frac{\sqrt{rrbb-rrcd}}{4}$. Wherefore we have found the following Rule to determine the *Focus* in the Ellipſis. *For if the fourth Part of the Figure* (or the Rectangle of the *Latus Rectum* into the *Latus Tranſverſum*) *be ſubſtracted from the Square of half the* Latus Tranſverſum, *and out of the Remainder you extract the ſquare Root, that will be the Diſtance* CN *of the Focus from the Center: And taking hence half the* Latus Tranſverſum CE, *you'll have* EN *the Diſtance of the Focus from the Vertex.*

SCHOLIUM I.

BOTH the Rules are eaſy in practice, for ſince $\frac{rrbb}{4}$ is nothing but the Square of CE, and $\frac{rrcd}{4}$ nothing but the Rectangle of ¼ DE into LM; if between LM and ¼ DE or MO (*Fig.* 125.) you find a mean Proportional MN, (whoſe □ therefore is equal to that ▭) and in the Hyperbola join to it at right Angles MC = CE, the Hypothenuſe CN will be the diſtance ſought of the *Focus* from the Center: And the ſame may be had in the Ellipſis, if (*N.* 2.) having deſcribed a Semi-circle upon CM = CE you draw or apply the Mean found MN and draw CN.

SCHOLIUM II.

HENCE alſo we have a new way of deſcribing the Ellipſis about two given conjugate Diameters A*a*, B*b*, (*Fig.* 126. *N.* 1, and 2.) Draw the right Line AH thro' A, one End of the given Diameter A*a*, perpendicular to the other Diameter B*b*, and take AQ in the ſame, on either ſide the Point A, equal to CB, and draw the Line CQ; then if the Line GF, equal to HQ, be moved ſo that the Ends thereof be always in the Lines B*b* CQ, (produced both ways from the Center, if neceſſary) till it has moved ſucceſſively thro' the four Angles formed by theſe Lines, and is come again to the Situation from which it went: I ſay, if GM be equal to AQ, that the Point M, by this motion, will deſcribe a Curve which will be an Ellipſis.

For draw GP parallel to QA, meeting the Diameter A*a* in P, and the Diameter B*b* in O; then becauſe the Triangles CHQ, COG, and CAQ, CPG are ſimilar, therefore CQ : CG :: AQ, or GM : GP :: HQ, or GF : GO; whence the Line PM will be parallel to the Diameter B*b*. This being ſuppoſed,

Call CA, *a*; AQ, or CB, *c*; and CP, *x*; PM, *y*; then will CA (*a*) : CP (*x*) :: AQ (*b*) : $\frac{bx}{a}$. And the right-angled Triangle GPM will give $\overline{PM}^2 = \overline{GM}^2 - \overline{GP}^2$; that is, $yy = bb - \frac{bbxx}{aa}$. Whence, by reſolving ~~this~~ Equation into an Analogy, it will be $\overline{AC}^2 (aa) : \overline{CB}^2 (bb) :: AP \times PA$ $(aa - xx) : \overline{PM}^2 (yy.)$ And conſequently the Line PM is an Ordinate to the Diameter A*a* in the Ellipſis, whoſe conjugate Diameters are the Lines A*a*, B*b*.

If the two conjugate Diameters A*a*, B*b*, be the Axis, then it is plain, that the Lines AQ, CQ, will fall in the Diameter A*a*, which will be one of the Axes, and the Point H will fall in the Center C. Therefore in this Caſe GF muſt be taken equal to the Sum or Difference of the two Semi-Axes, and the Ends thereof mov'd along the Axes.

But after what way the ſame Ellipſis may be deſcribed thro' infinite Points, will be manifeſt from the ſame Figures to any attentive Perſon. For the Angle QCH being once determin'd, if GF be apply'd where you pleaſe by help of a Pair of Compaſſes, and FM be taken equal to AH, you will have every where the Point K, *&c.*

CONSECTARY IV.

IN the Hyperbola, (*Fig.* 123.) ſince the □ of CN — □ of CE = ▭ DNE, and in the Ellipſis (*Fig.* 124.) □ of CE — □ of CN = ▭ DNE, by virtue of *Prop.* 9, and 8. *lib.* 1. if for CN you put on both ſides, for brevity's-ſake, *m*, then will the ▭ DNE in the Hyperbola be rightly expreſſed in theſe Terms, $mm - \frac{rrbb}{4}$; and in the Ellipſis, in theſe, $\frac{rrbb}{4} - mm$.

PROPOSITION VI.

IN the Parabola, the Latus Rectum is to the Sum of two Semi-ordinates (e. g. IK + FG, i. e. HO *in* Fig. 122.) *as their Difference* (OG) *is to the Difference of the Abſciſſæ* (IF *or* KO.)

Demonſtration.

For if the greater Abſciſſa EF be made $= ib$, and the leſſer EI $= sb$, the Semi-ordinates anſwering to them, FG and IK will be $\sqrt{rric}$ and $\sqrt{rscc}$, as is deduc'd from *Prop.* 1. Wherefore if you ſet in the ſame Series

$$\overset{1}{\textit{Latus R.}} : \overset{2}{\text{Sum of the Semiord.}} :: \overset{3}{\text{their Diff.}} : \overset{4}{\text{Diff. of Abſciſſ.}}$$

$$\frac{rcc}{b} : \sqrt{ricc} + \sqrt{rscc} :: \sqrt{ricc} - \sqrt{rscc} : ib - sb$$

and multiply the Extremes and Means; you'll have on both ſides the ſame Product $ricc - rscc$, which will prove the Proportionality of the ſaid Quantities, by virtue of *Prop.* 19. *lib.* 1. Q. E. D.

SCHOLIUM.

THIS is that Property of the Parabola, whereon the *Clavis Geometrica Catholica* of Mr. *Thomas Baker* is founded, which, as unknown to the Ancients, nor yet taken notice of by *Des Cartes*, he thinks was the reaſon why that incomparable Wit could not hit upon thoſe univerſal Rules for ſolving all Equations howſoever affected. Concerning which, we ſhall ſpeak further in its place. We will only further here note, that *Baker* was not the firſt Inventor of this Property, but had it, as he himſelf ingenuouſly confeſſes, out of a Manuſcript communicated to him by *Tho. Strode* of *Maperton*, Eſquire.

PROPOSITION VII.

IN the Hyperbola and Ellipſis, * *the Latus Rectum is to the Latus Tranſverſum, as the Square of any Semi-ordinate* (e. g. IK *in* Fig. 118, *and* 120.) *to the Rectangle* (DIE) *contained under the Lines intercepted between it and the Vertexes of the Latus Tranſverſum.*

* *The 3d Property of the Hyperb. and Ellip Apollon. lib.* 1. *Prop.* 21. *Part* 1.

Demonſtration.

For the *Latus Rectum* in both Sections is $\frac{rcd}{b}$, and the *Latus Tranſverſum* rb, &c. Wherefore if you make in the ſame Series the *Latus R.* : the *Lat. Tranſv.* :: □ of IK : ▭ DIE

$$\frac{rcd}{b} : rb :: \left\{ \begin{array}{l} \text{in Hyperb. } rscd + sscd : rsbb + ssbb \\ \text{in Ellipſ. } rscd - sscd : rsbb - ssbb \end{array} \right.$$

The Rectangles of the Extremes and Means will both be $rrsbcd \pm rssbcd$, and ſo will prove the Proportionality of the ſaid Quantities, by *Prop.* 19. *lib.* 1. Q. E. D.

CONSECTARY I.

HENCE in the Ellipſis, (ſee *Fig.* 124.) the *Latus Rectum* and the tranſverſe Axis being given, you may eaſily obtain the ſecond Axis, or Diameter if you make

$$\begin{array}{ccccccc} \textit{Lat. Tranſv.} & : & \textit{Lat. Rect.} & :: & ▭\,DCE & : & □\,AC \\ rb & : & \frac{rcd}{b} & :: & \frac{rrbb}{4} & : & \frac{rrcd}{4}. \end{array}$$

CONSECTARY II.

THerefore the □ of the whole AB will be $= rrcd =$ ▭ of the *Latus Rectum* into the *Latus Tranſv.* (which *Apollonius* calls the *Figure*;) ſo that the ſecond Axis (and any ſecond Diameter) will be a mean Proportional between the *Latus Rectum* and the *Latus Tranſverſum.* Hence, in the Hyperbola, alſo the ſecond or conjugate Diameter may be called a mean Proportional, between the *Latus Rectum* and *Tranſverſum*, i. e. $\sqrt{rrcd}$, or a Line which is equal in power to the *Figure*, as *Apollonius* ſpeaks.

SCHOLIUM I.

HENCE ariſes another and more eaſy way of delineating an Ellipſis *in Plano*, about the given Axes AB, DE (*Fig.* 127.) which *Schooten* has given us. Take two equal Rulers CG, GK, moveable at the Point G, like a Carpenter's Rule, and faſten the End C to the Center, ſo as to be moveable about it; and of ſuch a Length, that the Parts CF, HK being taken equal to the leſſer Axis AB, CF + FG + GH, may be equal to CD or CI, the half of the greater Axis DE. Then if the Point A be mov'd along the Line DE produced, the Point H will deſcribe the Curve EHAD. That this will be an Ellipſis, will be evident by virtue of this ſeventh *Prop.* from a Property that agrees to this Curve in all its Points H. For having drawn Circles about each Diameter, and the Lines IHN, FO, Perpendicular to CE; and having made the *Latus Rectum* EL, which is a third Proportional to DE and AB, by the ſecond *Conſect.* of this *Prop.* &c. by reaſon of the Similitude of the Triangles CFO, CIN, FO will be to FC, as IN to IC, and alternatively FO : IN : : FC : IC; *i. e.* as AC : CE : : AB : DE. Therefore alſo the Square of FO (or HN) will be to the Square of IN, as the Square of AB to the Square of DE, by *Prop.* 22. *lib.* 1. *i. e.* as EL the *Latus Rectum* to ED the *Latus Tranſverſum*, by *Prop.* 35. *lib.* 1. But the □ of IN is = DNE from the Property of the Circle. Therefore □ of FO (or of the Semi-ordinate HN) is to the □ DNE, as EL the *Latus Rectum* to ED the *Latus Tranſv.* Therefore by virtue of the preſ. *Prop.* the Point H is in the *Ellipſis*, and ſo any other, *&c.* Q. E. D.

CONSECTARY III.

NOW in the Ellipſis, if the □ of AC, the ſecond Axis ($=\frac{rrcd}{4}$, by *Conſect.* 1.) and □ of CN the Diſtance of the *Focus* from the Centre ($=\frac{rrbb - rrcd}{4}$, by *Conſect.* 3. *Prop.* 5. the Figure whereof you may ſee *N.* 124.) be joined in one Sum; the □ of AN will be $=\frac{rrbb}{4}$, and ſo the Line AN $=\frac{rb}{2}$, *i. e.* to half the *Latus Tranſverſum:* So that hence, having the Axes given, you may find the *Foci*, if from A at the interval CD you cut the tranſverſe Axis in N and *n*.

CONSECTARY IV.

NOW, on the contrary, in an Hyperbola, if (*Fig.* 123.) the □ of AC or EF $=\frac{rrcd}{4}$ be ſubſtracted from the □ of CF or CN $=\frac{rrbb + rrcd}{4}$, by virtue of *Conſect.* 2. *Prop.*

Prop. 5. there will remain $\frac{rrbb}{4}$, and its Root will be $\frac{rb}{2}$, *i. e.* half the *Latus Tranſverſum* CD: So that here alſo, the Axis being given, you may find the *Focus's*, if from the Vertex E you make EF a Perpendicular to the Axis = to the ſecond Axis AC, and at the Interval CF from the Centre C you cut the *Latus Tranſverſum* (continued) in N and *n*.

SCHOLIUM II.

BUT now, that the right Lines KN and K*n* drawn from any other Point (*e. g.* K.) to the *Foci*, when taken together in the Ellipſis; but when ſubſtracted the one from the other in the Hyperbola, are equal to the *Latus Tranſverſum* DE, we will a little after demonſtrate more univerſally, and alſo ſhew an eaſy and plain *Praxis* of delineating the Ellipſis and Hyperbola *in Plano*, having the Axes, and conſequently the *Foci* given.

CONSECTARY V.

SINCE we have before demonſtrated *Conſ.* 2, and 3. *Prop.* 5. that the ▭ DNE in the Hyperbola, and alſo in the Ellipſis, is $= \frac{rrcd}{4}$; and here in *Conſect.* 1. the □ of the ſecond Semi-diameter AC is alſo $= \frac{rrcd}{4}$; it is evident that this □ of AC is equal to the ▭ DNE.

CONSECTARY VI.

IT is hence moreover evident, if the Square of half the tranſverſe Diameter GE $= \frac{rrbb}{4}$ be compared with the Square of half the ſecond Diameter AC or EF $= \frac{rrcd}{4}$, multiplying both Sides by 4, and dividing by *r*; they will be to one another as *rbb* to *rcd*, *i. e.* further dividing both Sides by *b*, as *rb* to $\frac{rcd}{b}$, the *Latus Tranſverſum*, to the *Latus Rectum*.

CONSECTARY VII.

BUT ſince alſo the Rectangle DIE is to the Square of IK, as the *Latus Tranſverſum* to the *Latus Rectum*, by virtue of the preſent *Prop.* 7. the Square of CE the ſemi-tranſverſe, will be to the Square of AC the ſemi-ſecond Diameter (or, by the 5th *Conſect.* of this, to the Rectangle DNE) as the Rectangle DIE to the Square of IK.

CONSECTARY VIII.

YOU may therefore now have the □ of IK (which otherwiſe in the Hyperb. is $rscd + sscd$, in the Ellipſe $rscd - sscd$, by virtue of *Prop.* 2. and 3.) in other terms: If you make □ of CE : ▭ DNE :: ▭ DIE : a fourth.

that is, $\frac{rrbb}{4} : mm - \frac{rrbb}{4} :: rsbb + ssbb$: a fourth, in the Hyperbola, by virtue of *Conſect.* 4. *Prop.* 5. or $\frac{rrbb}{4} : mm - \frac{rrbb}{4} :: rsbb - ssbb$: a fourth in the Ellipſis.

From hence, by the Rule of Proportion, the Square of IK, being the fourth Proportional, will be $\frac{4smm}{r} + \frac{4ssmm}{rr} - rsbb - ssbb$ in the Hyp. and $rsbb - ssbb - \frac{4smm}{r} + \frac{4ssmm}{rr}$ in the Ellipſis, the Uſe of which Quantities will preſently appear.

PROPOSITION VIII.

THE $\left\{\begin{array}{l}\textit{Aggregate in the Ellipſis}\\ \textit{Difference in the Hyperb.}\end{array}\right\}$ *of the right Lines* * KN *and* K*n* (Fig. 128,) *drawn from the ſame Point* K *to both the Focus's, is equal to the tranſverſe Axis* DE.

* *Apollon. Lib.* 3. *Prop.* 51. *and* 52.

An Ocular Demonſtration;

WHich conſiſts wholly in this, *viz.* to find the Lines KN and K*n*, by help of the right-angled Triangles IKN and IK*n* (that is, the Hypothenuſes, by having the Sides given) and afterwards ſee if the Sum of both, in the Ellipſis, and Difference in the Hyperbola be $=rb$, *i.e.* to the tranſverſe Axis DE.

I. *In the Ellipſis.*

Putting m for CN (which above *Prop.* 5. *Conſ.* 3. was found to be $\frac{\sqrt{rrbb-rrcd}}{4}$) and then you'll have

$$\text{IN}=\text{CI}+\text{CN}=\tfrac{1}{2}rb-sb+m$$
$$\text{I}n=\text{C}n-\text{CI}=m-\tfrac{1}{2}rb+sb$$

Therefore □ of IN $=\frac{1}{4}rrbb-rsbb+ssbb+rbm-2sbm+mm$, and the □ of I$n=\frac{1}{4}rrbb-rsbb+ssbb-rbm+2sbm+mm$, Add to each the □ of IK, which was found in *Prop.* 7. *Conſect.* 8. in the Ellipſis $=rsbb-ssbb-\frac{4smm}{r}+\frac{4ssmm}{rr}$, and you will have □ of KN $=\frac{1}{4}rrbb+rbm-2sbm+mm-\frac{4smm}{r}+\frac{4ssmm}{rr}$, and the □ of K$n=\frac{1}{4}rrbb-rbm+2sbm+mm-\frac{4smm}{r}+\frac{4ssmm}{rr}$; and by extracting the Roots, (which is eaſy) you'll have

$$\text{KN}=\tfrac{1}{2}rb+m-\frac{2sm}{o} \text{ and}$$
$$\text{K}n=\tfrac{1}{2}rb-m+\frac{2sm}{o}$$

Sum rb. Q.E.D.

II. *In the Hyperbola.*

Putting m again for CN (which above, *Conſ.* 2. *Prop.* 5. was found to be $\frac{\sqrt{rrdc+rrbb}}{4}$) and you'll have

$$\text{IN}=\text{CI}+\text{CN}=\tfrac{1}{2}rb+sb+m$$
$$\text{I}n=\text{CI}-\text{C}n=\tfrac{1}{2}rb+sb-m. \text{ Therefore}$$

□ of IN $=\frac{1}{4}rrbb+rsbb+ssbb+rbm+2sbm+mm$; and
□ of I$n=\frac{1}{4}rrbb+rsbb+ssbb-rbm-2sbm+mm$

Add to both the □ of IK, which was found in *Prop.* 7. *Conſect.* 8. in the Hyperbola $\frac{4smm}{r}+\frac{4ssmm}{rr}-rsbb-ssbb$, and you'll have

□ of KN $=\frac{1}{4}rrbb+rbm+2sbm+mm+\frac{4smm}{r}+\frac{4ssmm}{rr}$

□ of K$n=\frac{1}{4}rrbb-rbm-2sbm+mm+\frac{4smm}{r}+\frac{4ssmm}{rr}$

and extracting the Roots out of theſe (which is eaſy) you'll have $KN = \frac{1}{2}rb + m + \frac{2sm}{r}$

$$Kn = \frac{1}{2}rb - m - \frac{2sm}{r}$$

(which is a negative and impoſſible Root; for it would be $CE - CN$: and moreover *minus* another Quantity.)

Or $Kn = m + \frac{2sm}{r} - \frac{1}{2}rb$; which is an affirmative and poſſible Root.

The difference therefore of the affirmative Roots is $= rb$. Q. E. D.

SCHOLIUM I.

WE firſt of all tried to make a literal Demonſtration, by uſing the Quantity of the Square IK, as you have it expreſſed, *Prop.* 2 and 3. and the Quantity IN as it was compounded of $CI = rb + sb + CN = \frac{\sqrt{rrcd + rrbb}}{4}$, &c. but we found it very tedious in making only the Squares of IN and I*n*: Then for the Surd Quantity CN we ſubſtituted another, *viz. m*, and we produced the Squares of IN and I*n* as above; but we added the Square of IK in its former value: and thus we obtain'd the Squares of KN and K*n*, but in ſuch terms, that the exact Roots could not be extracted, but muſt be exhibited as Surd Quantities, and conſequently we muſt make uſe of the Rules belonging to them, to find their Sum or Difference, which we laid down, *Conſ.* 3. *Prop.* 7. and *Conſect. Prop.* 10. *Lib.* 1. which tho' it would ſucceed, yet would be full of trouble and tediouſneſs. Therefore at length when we came to uſe thoſe other Terms which expreſs the Square of IK, the buſineſs ſucceeded as eaſy as we could wiſh, and that in a plain and eaſy way, and no leſs pleaſant; which I doubt not but will alſo be the opinion of the Reader, who ſhall compare this with other Demonſtrations of the ſame thing, which only lead indirectly to this truth, or with them which *de Witte* has given us in *Elem. Curvar. Lin. p. m.* 293. and 302. and which he thinks eaſy and ſhort enough in reſpect of others, both of the Antients and Moderns, and which we have reduced into this yet more diſtinct form, and accommodated to our Schemes.

Pre-

Preparation for the Hyperbola.

Make as $\left\{\begin{matrix}\text{CD : CN}\\ \text{CE : C}n\end{matrix}\right\}$ :: CI : CM

So that the ▭ of $\left\{\begin{matrix}\text{DCM}\\ \text{MCE}\end{matrix}\right\}$ will be = ▭ $\left\{\begin{matrix}\text{NCI}\\ n\text{CI}\end{matrix}\right.$

Becauſe therefore it will be, by *Conſect.* 7. *Prop.* 7.
as □ of CD : ▭ DNE :: ▭ DIE : □ of IK.

And alſo, by compoſition,

as □ of CD : $\left\{\begin{matrix}\square\text{ of CD} + \square\text{ of DNE}\\ \textit{i.e.}\ \square\text{ of CN, }\textit{per 9. lib.}\text{1.}\end{matrix}\right\}$ :: DIE : DIE + □ IK

Therefore by a Syllepſis,

as □ of CD : □ of CN :: $\left\{\begin{matrix}\square\text{ CD} + \text{DIE}\\ \textit{i.e.}\ \square\text{ of CI}\end{matrix}\right\}$: □ CN + DIE + □ IK

But alſo by the Hypotheſis,
as □ of CD : □ of CN :: □ of CI : □ of CM. Therefore,
□ of CM is = □ of CN + DIE + □ of IK.

Preparations for the Ellipſis.

Make as CD : CN :: CI : CM.
So that the ▭ DCM may be = ▭ NCI.
Becauſe therefore, by *Conſect.* 7. *Prop.* 7.
as □ of CD : ▭ DNE :: ▭ DIE : to □ of IK.

Therefore alſo by dividing,

as □ of CD : $\left\{\begin{matrix}\square\text{ of CD} - \text{DNE}\\ \textit{i. e.}\ \square\text{ of CN, by 8. }\textit{l.}\text{1.}\end{matrix}\right\}$ :: DIE : DIE − □ IK

Therefore by a Dialepſis,

as □ of CD to □ of CN, ſo $\left\{\begin{matrix}\square\text{ CD} - ▭\text{ DIE}\\ \textit{i.e.}\ \square\text{ CI, by 8. }\textit{cit.}\end{matrix}\right\}$ to □ of CN − DIE + □ IK

But alſo by the Hypotheſis,
as □ of CD : □ CN :: □ of CI : □ of CM
Therefore □ of CM is = □ of CN − DIE + □ of IK.

Demonſtration.

Since therefore it is certain, that the Difference or Sum of DM and EM is = to the tranſverſe Axis DE, if it be demonſtrated that DM is = KN and EM = K*n*, the buſineſs will be done, becauſe the Difference or Sum of KN and K*n* is alſo equal to the tranſverſe Axis DE.

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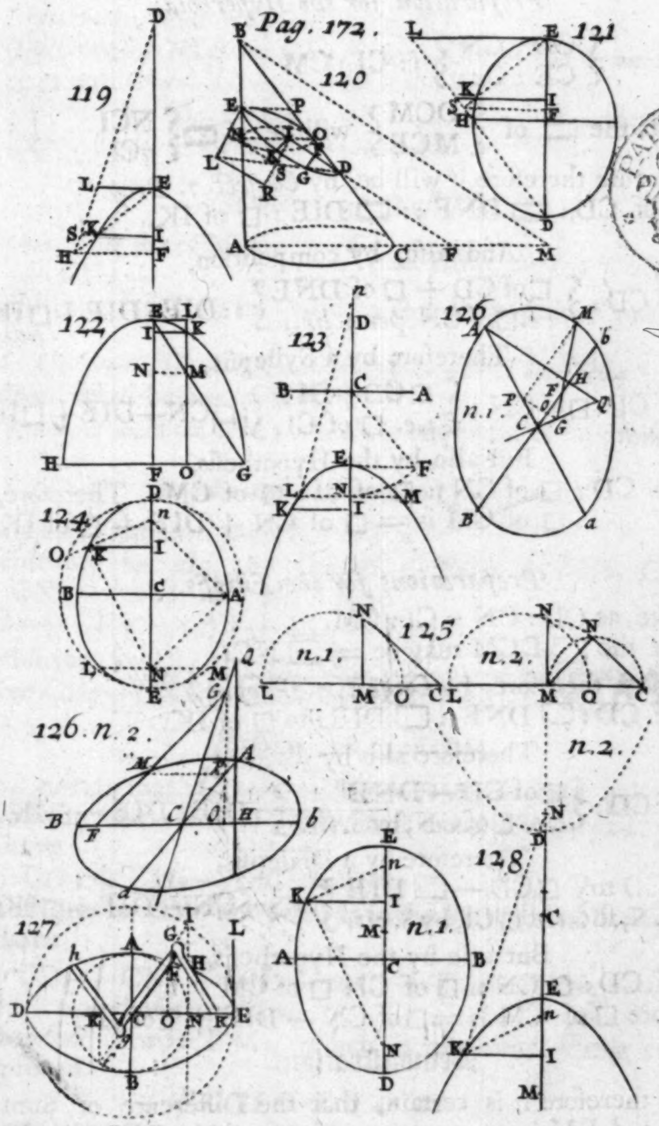

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Reſolve the □ *of* KN.

It is certain, that $NIq + IKq = KNq$
Subſtitute for NIq, by the 7. *lib.* 1. $CIq + CNq + 2\,NCI$
Then will $CIq + CNq + 2\,NCI + IKq = KNq$.
Subſtitute for CIq, by the 9. *lib.* 1. $CDq + DIE$; then will
$CDq \pm DIE + CNq + 2NCI + IKq = KNq$.

Reſolve alſo □ *of* DM.

It is certain that $CMq + CDq + \left\{ \begin{array}{c} 2DCM \\ 2NCI \end{array} \right\} = DMq$, by the 7. *lib.* 1.
Subſtitute for CMq its value by the Preparation, and you'll have

$$CNq - DIE + IKq + CDq + 2NCI = DMq$$

Which were before $= KNq$
Therefore $KN = DM$; which is one thing to be proved.

In like manner reſolve □ *of* K*n*.

It is certain that $nIq + IKq = Kn$
Subſtitute for nIq, by *Conſect.* 1. *Prop.* 10. *lib.* 1. $CIq + CNq - 2nCI$, and you'll have
$CIq + CNq - 2nCI + IKq = Knq$.
Subſtitute for CIq, by the 9. *lib.* 1. $CDq \mp DIE$, and you'll have $CDq \pm DIE + CNq - 2nCI + IKq = Knq$

Reſolve alſo □ *of* EM.

It is certain that $2CDq + 2CMq - DMq = EMq$, *per* 13. 1.
Subſtitute the value of DMq, firſt found above, and you'll have

$$CDq + CMq - 2nCI = EMq$$

Subſtitute for CMq the value as in the Preparation, and you'll have

$$CDq + CNq \mp DIE - 2nCI + IKq = EMq$$

Which were before $= Knq$
Therefore $Kn = EM$; which is the other thing to be proved.

SCHOLIUM II.

HEnce you have the common mechanical Ways of deſcribing the Ellipſis and Hyperbola about their given Axes; for if the *Foci* N, *n* (*Fig.* 129. *N.* 1.) are given or found, according to *Conſect.* 3. *Prop.* 7. and having therein ſtuck or fixed two Pins, put over them a Thread NF*n*, tyed both ends together, preciſely of the Length you deſign the greater Axis DE to be of; and having put your Pencil or Pen in that triangular String, draw it round, always keeping it equally extended or tight. Now becauſe the Parts or Portions of the Thread remain always equal to the whole Axis DE, what we propoſed is evident by the preſent *Prop.* which may alſo be very elegantly deſcribed by a certain ſort of Compaſſes, a Deſcription whereof *Swenterus* gives us in his *Delic. Phyſico-Math. Part* 2. *Prop.* 20. which may be alſo done by a ſort of organical Mechaniſm, by help of two Rulers GN, H*n*, moveable in the *Foci* (*n.* 2.) in length equal to the tranſverſe Axis DE, and faſtened together by a croſs Ruler GH, equal in length to the diſtance of the *Foci*, as may appear from the Figure. For if the Pin F be moved round in the Interſection of the croſs Rulers H*n* and GN, the Curve thereby deſcribed will be an Ellipſis, from the Property we have juſt now demonſtrated of it, which it hath in every Point F. For the Triangles HGN and NH*n*, which have one common Side HN, and the others equal, by conſtruction, are equal one to another, and conſequently the Angles FHN and FNH equal, ſo alſo the Legs HF and FN, and ſo likewiſe FN and F*n* together are equal to H*n* = DE; which is the very Property of the Ellipſis we are now treating of. But *Van Schooten*, who taught us this Delineation, hints, that, if thro' the middle Point I, of the Line HN, you draw the Line IFL, it will touch the Ellipſis in the Point F; for ſince the Angles IFH and IFN are equal, by what we have juſt now ſaid, the vertical Angle LF*n* of the one IFH, will be neceſſarily equal to the other IFN: But this Equality of the Angles, made by the Line KL drawn thro' F, with both thoſe drawn from the Centers, is here a ſign of contact, as is in the Circle the Equality of the Angles, with a Line drawn from its Centre. So that after this way you may draw a Tangent thro' any given Point F of the Ellipſis, without this organical *Apparatus* of Rulers; *viz.* if, having drawn from both the *Focus's*, thro' the given Point F, the right Lines *n*H, NG equal to the *Latus Tranſverſum* DE, you biſect HN in I, and draw IFL: or if the Line that connects the Extremes GH be produced to K, and you draw thence KFL, *viz.* in that caſe where GH and N*n* are not parallel; otherwiſe a Line drawn through the Point F parallel to them would be the Tangent ſought.

As to the Hyperbola, there is a mechanick Method of drawing that alſo, not unlike the others, from a like Property in that, communicated by the ſame *Van Schooten*; for having found the *Focus's* N and *n*, (*Fig.* 129. *N.* 3.) tie a Thread NFO in the *Focus* N, and at the End of the Ruler *n*O of the Length of the tranſverſe Axis DE; then putting in a Pen, or the moveable Leg of a Pair of Compaſſes, (nor would it be difficult to accomodate the Practice we before made uſe of to this alſo) draw or move it within the Thread NFO, from O to E, ſo that the part of the Thread NO may always keep cloſe to the Ruler, as if it were glued to it. For if we call the Length of the Thread X, and the tranſverſe Axis *rb* as above, the Ruler *n*O will be, by the Hypoth. $= X + rb$. Make now the Part of the Thread $OF = \frac{1}{2}X$; then the Remainder or other Part will be $NF = \frac{1}{2}X$, and $nF = \frac{1}{2}X + rb$, and the difference between FN and F*n*, $= rb$. Make $OF = \frac{3}{4}X$, then will FN be $\frac{1}{2}X$ and $Fn = \frac{1}{4}X + rb$, the Difference ſtill remaining *rb*, and ſo *ad infinitum.* In ſhort, ſince the Difference of the whole Thread, and of the whole Ruler is *rb*, and in drawing them, the ſame OF is taken from both, there will always be the ſame Difference of the Remainders. Hence alſo, aſſuming at pleaſure the Points N and *n*, you may deſcribe Hyperbola's, ſo the Thread NFO be ſhorter than the Ruler *n*FO: For if it were equal, there would be deſcribed a right Line perpendicular to N*n*, thro' the middle Point C.

There yet remains one Method of deſcribing Hyperbola's and Ellipſes *in Plano*, by finding many Points, without the Help or *Apparatus* of any Threads or Inſtruments, *viz.* in the Ellipſis, having given or aſſumed the tranſverſe Axis DE, and the *Foci* N and *n* (*Fig.* 130, *N.* 1.) if from N at any arbitrary Diſtance, but not greater than half the tranſverſe Axis NF, you make an Arch, and keeping the ſame opening of the Compaſſes, you cut off EG from the tranſverſe Axis; and then, taking the remaining Interval GD from *n*, you make another Arch, cutting the former in F, and ſo you will have one Point of the Ellipſis; and after the ſame way you may have innumerable others, *f*, *f*, *f*, *&c.*

In like manner, to delineate the Hyperbola, having given or aſſumed the tranſverſe Axis DE, and the *Focus's* N and *n* (*N.* 2.) if from N, at any arbitrary Diſtance NF, you ſtrike an Arch, and keeping the ſame opening of the Compaſſes from the Diameter continued, you cut off EG, and then at the Interval GD from *n* make another Arch cutting the former in F, you will have one Point of the Hyperbola, and after the ſame way innumerable others, *f, f, &c.*

PROPOSITION IX.

IF the ſecundary Axis, or conjugate Diameter of the Hyperbola AB (Fig. 131.) *be applied to the Vertex* E, *ſo that it may touch the Hyperbola, and* OE, EP *are equal, like as* BC *and* AC, *and if from the Centre* C *you draw right Lines thro'* O *and* P *running on* ad infinitum, *and, laſtly,* QR *parallel to the Tangent* OP; *you will have the following*

CONSECTARIES.

I. THE Parts QG and HR intercepted between the Curve and thoſe right Lines CQ, CR will be equal; for (by reaſon of the Similitude of the Triangles CEP and CFR, as alſo CEO, CFQ) as CE is to EO (and EP) ſo will CF be to FQ and FR, and conſequently theſe will be equal; and ſo taking away the Semi-ordinates FG and FH, which are alſo equal, the Remainders GQ and HR will be alſo equal, and conſequently the Rectangles QGR, GRH, *&c.* all equal among themſelves: which we had already ſhewn in one particular Caſe before in *Conſ.* 2, and 3. *Def.* 7.

II. The Rectangle QGR will be = □ of EO or EP $= \frac{rrcd}{4}$, i. e. (as *Apollonius* ſpeaks) to the fourth Part of the Figure: For by reaſon of the Similitude of the Triangles CEO, CFQ, CE will be to EO as CF to FQ; *i. e.*

as the □ of CE to the □ of EO
i. e. as (by *Conſ.* 2, 7.) the *Lat. Tranſv.* to the *Lat. Rect.*
i. e. (by the 7th *Prop.*) as the □ DFE to the □ of FG
} ſo the □ of CF to the □ of FQ.

But now if from the □ of CF you take the Rectangle DFE, there will remain the □ of CE, by *Prop.* 9. *lib.* 1. and from the □ FQ you take the Rectangle FG, there will remain the Rectangle QGR, by *Prop.* 8. *lib.* 1. wherefore that remaining □ of CE to this remaining Rectangle QGR, will be as the whole Square of CF to the whole Square of FQ, by *Prop.* 26. *lib.* 1. *i. e.* as the □ of CE to the □ of EO; conſequently the Rectangle QGR, and the Square of EO (to which the ſame Square of CE bears the ſame Proportion) will be equal among themſelves.

III. Since this is alſo after the ſame manner certain of any other Rectangle *qgr* or *grh*, *&c.* it follows, that all ſuch Rectangles are equal among themſelves.

IV. Wherefore ſince the Lines FR, *fr*, *&c.* and ſo GR and *gr* grow ſo much the longer, by how much the more remote they are from the Vertex E; it is evident, that the Lines QG and *qg* muſt neceſſarily, on the contrary, ſo much the more decreaſe and grow ſhorter, and conſequently the right Line CQ approach ſo much nearer and nearer to the Curve EG.

V. But that they can never meet or coincide, altho' produced *ad infinitum*, will thus appear; if it were poſſible there could be any concurrence or meeting, ſo that the Point G and Q, or *g* and *q* could any where coincide, it would follow, from *Conſect.* 2. that as the Rectangle DFE to the Square FG, ſo would the Square of CF to the Square of FQ, *i.e.* to the ſame Square of FG; and ſo that the Rectangle DFE would be $= \square$ CF; which is abſurd by *Prop.* 9. *lib.* 1. ſo that now it is evident, that the Lines COQ and CPR, drawn according to *Conſect.* 1. are really Aſymptotes, *i. e.* they will never * coincide (*viz.* with the Curve of the Hyperbola) as *Apollonius* has named them.

* *Apoll. Prop.* 1. *lib.* ii.

VI. Having drawn right Lines from G and *g* parallel to both the Aſymptotes, *viz.* GS and *gs*, and likewiſe GT and *gt*; the Rectangles TGS and *tgs* will be † equal among themſelves. For by reaſon of the Similitude of the Rectangles TQG and *tqg*, TG will be to QG as *tg* to *qg*; and, by reaſon of the Equality of the Rectangles QGR and *qgr*, QG will be to *gr* reciprocally as *qg* to GR, by *Prop.* 19. *lib.* 1. and by reaſon of the Similitude of the Triangles SGR and *sgr*, as *gr* to *gs*, ſo GR to GS: wherefore (ſince

$$\begin{array}{l} TG : QG :: gr : gs \\ tg : qg :: GR : GS) \end{array}$$

you will have, by proportion of Equality, as TG to *gs*, ſo *tg* to GS, by *Prop.* 24. *lib.* 1. Therefore, by *Prop.* 18. of the ſame, the ▭ of TG into GS = ▭ of *tg* into *gs*. Q. E. D.

† *Apoll. Prop.* 12. *lib.* ii.

SCHOLIUM.

HENCE, laſtly, we have a new Geneſis of the Hyperbola *in Plano*, about its given Diameters, from the Speculations of * *De Witt*: Draw the Lines AB, EF, cutting one another at pleaſure

* *De Witt. Elem. Curv. lib.* i. *cap.* 1. *Prop.* 3.

(*Fig.* 132.) and to the Angle BCF conform the moveable Angle BCD (in delineating the oppoſite Hyperbola *acd*, being equal to the contiguous Angle ACD) having one Leg thereof indefinitely extended, and the other CD of any Length at pleaſure, and to the End D thereof apply the Slit of a Ruler GD moveable about the Point G taken any where at pleaſure, ſo that GD be parallel to the Leg CB in this firſt Station. Then if you turn the Ruler GD about the fixed Point G, and this carries the Leg CD of the moveable Angle BCD along the Line ECF, the Interſection of the Ruler GDH, and the other Side CB of the Angle (*b* and β being two of them) will deſcribe the Curve *b*Gβ, which we thus prove to be an Hyperbola: Becauſe the Ruler GDH turning about the Pole G, and carried from D, *e. g.* to *d* or δ cuts the Leg of the moveable Angle CB brought to the Situation *cb* or γβ, and in the mean while remaining always parallel to itſelf, drawing from the Points of Interſection *b*, β, and G, the Lines GI, *b*K, and βϰ, parallel to the Ruler CF; becauſe, *e. g.* in the ſecond Station, having taken the common Quantity *c*D from the equal ones CD and *cd*, the Remainders C*c* and D*d* are equal, and by reaſon of the Similitude of the Triangles *dcb* and *d*DG,

$$\left.\begin{array}{l}\text{as } Dd\\ i.e.\ Cc\\ \text{or } bK\end{array}\right\} : DG :: \left\{\begin{array}{c} dc\\ i.e.\ DC\\ \text{or } GI\end{array}\right\} : cb;$$

the Rectangle of K*b* into *bc*, will be = ▭ of DG into GI, by *Prop.* 18. *lib.* 1. And in like manner, when, in the third Station, having added the common Line Dγ to the equal ones CD and γδ, the whole Lines Dδ and Cγ are equal, and, by reaſon of the Similitude of the Triangles βγδ and GDδ,

$$\left.\begin{array}{l}\text{as } D\delta\\ i.e.\ C\gamma\\ \text{or } \beta\varkappa\end{array}\right\} : DG :: \left\{\begin{array}{c} \gamma\delta\\ i.e.\ DC\\ \text{or } GI\end{array}\right\} : \gamma\beta;$$

the ▭ of ϰβ into βγ = ▭ of DG into GI, by the ſame 18th *Prop.* Wherefore the three Points *b*, G, β, (and ſo all the others that may be determined the ſame way) are in the Hyperbola, whoſe Aſymptotes are CB and CF, and its Center C, &c. by the preſent *Prop. Conſect.* 6. Q. E. D.

You may alſo determine innumerable Points of this Curve ſeparately, without the Motion we have now preſcribed; as the Point *a* in the oppoſite Hyperbola, if thro' any Point *c* taken in the Aſymptote CE, you draw a Parallel to the other Aſymptote CA, and having made *cd* equal to CD, from G thro' *d*, you draw G*da*, and ſo in others.

CHAP. II.

Of Parabolick, Hyperbolick, and Elliptick Spaces.

PROPOSITION X.

THE * *parabolick Space* (vide Fig. 133.) *that is the Figure comprehended under the right Line* GH *and the Parabola* GEH, *is to a circumſcribing Parallelogram* GK, *as* 4 *to* 6 (*or* 2 *to* 3) *but to an inſcribed* △ GEH, *as* 4 *to* 3.

Demonſtration.

Suppoſe FH divided firſt into two, then into four equal Parts, and draw the Lines *ef*, *ef*, &c. parallel to the Axis EF; and ſuppoſing EF to be 4, then the firſt *fg* will be 3, the ſecond 2, and the third 1, by *Prop.* 34. *lib.* 1. but as *ef* is to *ge*, ſo is *ge* to *he*, by *Conſect.* 1. *Prop.* 4. Therefore *he* in the Diameter FF is $= o$, in the firſt *ef* it is $=\frac{1}{4}$ (for as *ef*, 4, to *ge*, 1, ſo is *ge*, 1, to *he*, $\frac{1}{4}$) in the ſecond *ef*, the Portion *he* is $=\frac{4}{4}$, in the third $=\frac{9}{4}$, and ſo the Portions *eh*, in the trilinear Figure E*h*HK, make a Series in a duplicate arithmetical Progreſſion, *viz.* 1, 4, 9, 16: After the ſame manner, if the Parts F*f*, &c. are biſected, you'll find the Portions *eh* in the external trilinear Figure to make this Series of Numbers $\frac{1}{8}$, $\frac{4}{8}$ $\frac{9}{8}$, $\frac{16}{8}$, $\frac{25}{8}$, $\frac{36}{8}$, $\frac{49}{8}$, $\frac{64}{8}$, and ſo onwards. Wherefore ſince the Portions *eh* or the Indiviſibles of the trilinear Space circumſcribed about the Parabola, are always in a duplicate arithmetical Progreſſion: the Sum of them all will be to the Sum of as many Indiviſibles of the Parallelogram FK, equal to the Line KH, *i. e.* the trilinear Space itſelf to this Parallelogram, as 1 to 3, by *Conſect.* 10. *Prop.* 21. *lib.* 1. Wherefore the Semi-parabola FE*h*H will be as 2, and the △ FEH as $1\frac{1}{3}$; therefore the whole Parabola as 4, and the whole △ GEH as 3, and the whole Parallelogram GK as 6. Q. E. D.

* *Archim. de Quadratur. Parab. Prop.* 17, *and* 24.

CONSECTARY I.

* *Archim. Prop.* 19. *with the Coroll.*

IT is evident * that in the firſt Diviſion, the ſecond Line *fh*, or the Line drawn from the middle of the Baſe FH, is three ſuch Parts, whereof FE is 4; for *eh* is $\frac{4}{4}$, *i.e.* 1, therefore *fh* is 3.

CONSECTARY II.

IT is alſo evident, that this Demonſtration will hold of any parabolick Segment.

PROPOSITION XI.

† *Archim. lib. de Conoid. &c. Prop.* 5.

THE elliptick Space † comprehended by the Ellipſis DAEB (Fig. 127.) *is to a Circle deſcribed on the tranſverſe Axis* DE, *as the Axis Rectus, or conjugate Diameter* AB, *to the tranſverſe Axis* DE.

Demonſtration.

THIS is in the firſt place evident from the Geneſis of the Ellipſis we laid down in *Schol.* 1. *Prop.* 7. for there we ſhewed that FO, *i.e.* HN was to NI, as AB to DE: which ſince it is true of all other Indiviſibles or Ordinates HN and IN *ad infinitum*; it is manifeſt, that the Planes themſelves conſtituted of theſe Indiviſibles will have the ſame Ratio among themſelves, as the *Axis Rectus* AB to the tranſverſe Axis DE. Q. E. D.

CONSECTARY I.

THerefore the Quadrature of the Ellipſis will be evident, if that of the Circle can be demonſtrated.

CONSECTARY II.

SINCE a Circle deſcribed on the leſſer Diameter AB will be to one deſcribed on the greater Diameter DE, as AB to a third Proportional, by *Prop.* 35. *lib.* 1. it follows, by virtue of the preſent *Prop.* that the Ellipſis is a mean Proportional between the greater and leſſer Circle, *i.e.* as the Ellipſis is to the greater Circle, ſo is the leſſer Circle to it, *viz.* as AB to DE.

CON-

CONSECTARY III.

HENCE you may have two ways of determining the Area of an Ellipſis. 1. Having found the Area of the greater Circle, ſay as the greater Diameter of the Ellipſis to the leſs, ſo is the Area of the Circle found, to the Area of the Ellipſis ſought. 2. Or having alſo found the Area of the leſſer Circle, find a mean Proportional between that and the Area of the greater Circle, and this will be the Area of the Ellipſis.

SCHOLIUM.

WE may alſo ſhew the laſt part of the ſecond *Conſect.* thus: If the Circle E*adb*E (*Fig.* 134.) be deſcribed about the leaſt Axis of the Ellipſis, and you conceive a regular Hexagon to be inſcribed in it, and an Ellipſis to coincide with one End E of its tranſverſe Axis, and the other or oppoſite one D to be ſo elevated, that it may perpendicularly hang over the Point D of the Circle, and further from all the Angles of the Figure inſcribed in the Circle, erect the Perpendiculars *g*G, *b*B, &c. then it is certain, that the Sides ED and E*d* of the Triangle DE*d* will be cut by the parallel Planes FG, *fg*, &c. into proportional Parts, and that thoſe, by reaſon of the Similitude of the Triangles FDG and *fdg*, will be among themſelves, and ſo alſo the other Rectangles, as the intercepted Parts ID, *id*, CI, *ci*, of the Lines, *in infinitum*, let the inſcribed Figure conſiſt of any number of Sides ſoever: Wherefore alſo all the Parts of the Ellipſis taken together, will be to all the Parts of the Circle taken together, *i. e.* the whole Ellipſis to the whole Circle, as all the Parts of the Diameter ED, or *ab*, *i. e.* as DE itſelf to AB. Q. E. D.

CONSECTARY IV.

IT is alſo evident, that both theſe Demonſtrations of the preſent *Prop.* will be alſo the ſame in any Segments of the Ellipſis or Circle.

PROPOSITION XII.

ANY hyperbolical Space GEHG (Fig. 135.) *is to any hyperbolick Figure of equal Height* gE*hg* [*whoſe Latus Rectum and Tranſverſum are equal (as in the Circle) and alſo equal to the Latus Tranſverſum* DE *of the former Figure*] *as the conjugate Axis* AB *is to the Latus Tranſverſum* DE, *as in the Ellipſis.*

Demonſtration.

By the Hypoth. and *Prop.* 7. and its ſecond *Conſect.* the □ of Fg is = ▭ DFE. Wherefore this ▭ DFE, *i. e.* the □ of Fg is to the □ of FG, as the *Latus Tranſverſum* to the *Latus Rectum* of the Hyperbola GEHG, by the ſame ſeventh *Prop. i. e.* (by *Conſ.* 2. of the ſame) as the Square of the *Latus Tranſverſ.* DE to the Square of the conjugate Axis AB: Therefore the Roots of theſe Squares will be alſo proportional, *viz.* Fg to FG, as DE to AB; and conſequently (ſince the ſame is true of any other Ordinates, *ad infinitum*) the whole Hyperbola gEhg will be to the whole one GEHG, as DE to AB. Q. E. D.

CONSECTARY I.

THerefore having found the Quadrature of ſuch an Hyperbola, whoſe *Latus Rectum* and *Tranſverſum* are equal, you may have alſo the Quadrature of any other Hyperbola.

CONSECTARY II.

IT is evident, that the ſame Demonſtration will hold in any other Hyperbola's.

PROPOSITION XIII.

ANY parabolick Segments upon the ſame Baſe, and hyperbolick and elliptick ones deſcribed about the ſame conjugate Diameter, (one whereof ſhall be a right Segment, *the other a* ſcalene one) *and conſtituted between the ſame Parallels, are equal.*

Demonſtration.

I. It is evident of Parabola's; for both the right one
p.196. GEHG, and the ſcalene one G𝔈HG (*Fig.* 136. *N.* 1.) becauſe the Demonſtration of *Prop.* 10. will hold in both, is to a △ inſcribed in them, as 4 to 3. But the Triangles GEH and G𝔈H are equal, by *Conſect.* 5. *Def.* 12. or *Prop.* 28. *lib.* 1. Therefore the Parabola's alſo.

Or thus, in the right Parabola GEHG, every thing is the ſame, as in *Prop.* 1, and 4. of this Book, *viz.* EI = sb, EF = ib, the Square of IK = $rscc$, the □ of FG = $ricc$. And therefore becauſe in the ſcalene Parabola alſo the Square of FG remains = $ricc$, make F𝔈 = n, and find both the Abſciſſæ 𝔈𝔍, and the □ of 𝔍𝔎 anſwering to it.

1. For the Abſciſſa; as FE : EI :: F$\mathfrak{E}$: $\mathfrak{E}\mathfrak{I}$, (*per Conſect.* 4. *Prop.* 34. *lib.* 1.) $ib : sb :: n : \frac{sn}{i}$.

2. For the □ $\mathfrak{I}\mathfrak{K}$; as F$\mathfrak{E}$: $\mathfrak{E}\mathfrak{I}$ :: □ of FG : □ of $\mathfrak{I}\mathfrak{K}$, (*per Prop.* 4. of this.) $n : \frac{sn}{i} :: ricc : rscc$.

Therefore the □ of $\mathfrak{I}\mathfrak{K}$ = □ of IK, and $\mathfrak{I}\mathfrak{K}$ = IK, and this in any caſe, *ad infinitum*: Therefore the one Parabola is = to the other. Q. E. D.

II. The buſineſs is evident much after the ſame way in Ellipſes and Hyperbola's: For making all things in the Ellipſis and right Hyperbola (*N.* 2, and 3. *Fig.* 136.) as in p. 196. *Prop.* 2, 3, 5, 7. *viz.* the □ of IK $rscd - sscd$ in the Ellipſis, $rscd + sscd$ in the Hyperbola, the □ of AB $= rrcd$, by *Conſect.* 2. *Prop.* 7. EI $= sb$, DE $= rb$, &c. if in oblique 58. ones for the *Latus Tranſverſum* $\mathfrak{D}\mathfrak{E}$ you put n, and ſeek the *Latus Rectum* and Abſciſſa $\mathfrak{E}\mathfrak{I}$, you may by means of theſe alſo have the Square of $\mathfrak{I}\mathfrak{K}$, by *Prop.* 2, and 3. 55.

1. *For the Latus Rectum.*

$$\text{As } n : \sqrt{rrcd} :: \sqrt{rrcd} : \frac{rrcd}{n}, \text{ by } Conſect.\ 2,\ \text{VII.}$$

2. *For* $\mathfrak{E}\mathfrak{I}$ *the Abſciſſa.*

$$\text{As } rb : sb :: n : \frac{sn}{r} = \mathfrak{E}\mathfrak{I}.$$

3. *For the Side* RS *of the deficient or exceeding Rectangle, from* Prop. 2, *and* 3.

$$\text{As } n : \frac{rrcd}{n} :: \frac{sn}{r} : \frac{rscd}{n} = \text{RS}.$$

Now the Abſciſſa multiplied by the *Lat Rect.*	The Abſciſſa multiplied by RS.
$\frac{sn}{r}$ by $\frac{rrcd}{n}$ gives ▭ $rscd$.	$\frac{n}{r}$ by $\frac{rscd}{n}$ gives ▭ $sscd$.
The Sum of theſe Rectangles $rscd + sscd$ in the Hyperb. = □ of $\mathfrak{I}\mathfrak{K}$, by *Prop.* 2. evidently = □ of IK.	The Difference of theſe Rectangles $rscd - sscd$ gives, in the Ellipſis □ of $\mathfrak{I}\mathfrak{K}$, by *Prop.* 3. evidently = □ of IK.

Wherefore the Lines 𝔍𝔎 and IK, and the whole Lines 𝔎𝔏 and KL will be equal; and ſince the ſame thing is evident after the ſame way of all other Lines of this kind *ad infinitum*, the elliptick and hyperbolick Segments will be ſo alſo. Q. E. D.

* * *

CHAP. III.

Of Conoids and Spheroids.

PROPOSITION XIV.

* *Archim. Prop.* 23, *and* 24. (*al.* 26, *and* 27.)

*A Parabolick Conoid * is ½ of a Cylinder, and in a ſeſquialteran Ratio, to a Cone, of the ſame Baſe and Altitude.*

Demonſtration.

p.196. Becauſe in the Parabola the □ of AD (*Fig.* 137.) is to the □ of SH, as BD to BH, *i. e.* as 3 to 1, and ſo to the
160. □ of TI, as BD to BI, *i. e.* as 3 to 2, by *Prop.* 4 of this; It is evident that theſe Squares of SH and TI, and AD, and conſequently of the whole Lines S*h*, T*i*, AC, and the Circles anſwering to them, will be in the arithmetical Progreſſion, 1, 2, 3; and moreover, if there are new Biſections, *in infinitum*, as the Abſciſſæ, ſo alſo the Squares and Circles, whoſe Diameters are the Ordinates, by virtue of the
ib. aforeſaid fourth *Prop.* will always be in the arithmetical Progreſſion 1, 2, 3, 4, 5, 6, &c. Whence it is evident, that an infinite Series of Circles in the Conoid, conſider'd as its Indiviſibles, will be to a Series of as many Circles equal to the greateſt AC, *i. e.* the Conoid to the Cylinder AF,
78. will be as 1 to 2, or as 1½ to 3, by *Conſect.* 9. *Prop.* 21.
65. or *Conſect.* 4. *Prop.* 16. *lib.* 1. but to the ſame Cylinder
109. AF, the inſcribed Cone ABC, is as 1 to 3, by *Prop.* 38. *lib.* 1. therefore the Cylinder, Conoid, and Cone are as 3, 1½, and 1. Q. E. D.

PRO-

PROPOSITION XV.

*THE half of * any Spheroid, or any other Segment of it, is $\frac{2}{3}$ of the Cylinder, and double of the Cone, having the ſame Baſe and Altitude.*

* *Archim.* 29, *and* 30. (*al.* 32, *and* 33.)

Demonſtration.

Having divided the Altitude BD (*Fig.* 138.) *e. g.* into three equal Parts, becauſe in the Ellipſis, as well as in the Circle, the Square of AD is to the Square of SH, as the Rectangle GDB to the Rectangle GHB, *i.e.* as 9 to 5; and ſo to the Square of TI as 9 to 8, by *Conſect.* 1. *Prop.* 5. of this: And in like manner, if you make new Biſections, the Squares (and conſequently the Circles) of the Ordinates go on or decreaſe by a Progreſſion of odd Numbers, as 36, 35, 32, 27, 20, 11, and ſo *ad infinitum*, the Biſections being continued on; as we have ſhewn in the Sphere and circumſcribed Cylinder, *Prop.* 39. *lib.* 1. and it will neceſſarily follow here alſo (by virtue of *Conſect.* 12. *Prop.* 21.) that the whole Cylinder will be to the inſcribed Segment of the Spheroid, as 3 to 2; and ſince the ſame Cylinder is to the Cone ABC, as 3 to 1, alſo the Segment of the Spheroid will be to the Cone as 2 to 1. Q. E. D.

p. 196. 161. 110. 80.

PROPOSITION XVI.

*AN Hyperbolick Conoid * is to a Cone of the ſame Baſe and Altitude, as the Aggregate of the Axis of the Hyperbola that forms it, and half the Latus Tranſverſum, to the Aggregate of the ſaid Axis and Latus Tranſverſum.*

* *Archim. Pro.* 27, *and* 28. (*al.* 30, *and* 31.)

Demonſtration, *containing alſo the Invention of this Proportion.*

Make (in *Fig.* 139.) CE $= a$, EF $= b$, and OE $= c$; then will CF $= a + b$. Since therefore

CE : OE :: CF : FQ

$$a : c :: a+b : \frac{ac+bc}{a}$$

the □ of EO will $= cc$, and □ of FQ $= \frac{aacc+2abcc+bbcc}{aa}$

But as theſe Squares, ſo alſo are the Circles of the Lines EO and FQ to one another, by *Prop.* 32. *lib.* 1. and ſo the Cone

196. 88.

Cone COP will be, as $\frac{acc}{3}$, and the Cone CQR, as $\frac{acc}{3}+bcc+\frac{bbcc}{a}+\frac{b^3cc}{3aa}$ (*viz.* by multiplying the third part of the Altitude CF by the Baſe FQ:) Having therefore ſubſtracted the Cone COP from the Cone CQR, there will remain the truncated Cone QOPR $= bcc+\frac{bbcc}{a}+\frac{b^3cc}{3aa}$, and from this ſolid truncated Cone having further ſubſtracted the hollow truncated Cone, which the Space EHRP produced in the Geneſis of the Conoid, (and which, according to *Conſect.* 2.
146. *Definit.* 9. is as bcc) there will remain the hyperbolick Conoid $\frac{bbcc}{a}+\frac{b^3cc}{3aa}$, *i. e.* (by ſubſtituting now the Values of the Axis or Abſciſſa EF, and of half the *Lat. Tranſv.* EC, and of the conjugate Diam. OP, &c. found in the Demonſtrations of the preceding Chapter, *viz.* $\frac{rb}{2}$ for a, sb for b, and $\sqrt{rrcd}$ for c, or $rrcd$ for cc) the Hyperbolick Conoid will come out $2rssbcd+\frac{4s^3bcd}{3}$, *i. e.* $\frac{6rssbcd+4s^3bcd}{3}$. But the Cone GEH (multiplying the third part of EF into the □ of GH, *i. e.* $\frac{1}{3}sb$ into $4rscd+4sscd$) is as $\frac{4rssbcd+4s^3bcd}{3}$. Therefore the Conoid is to the Cone as $6rssbcd+4s^3bcd$ to $4rssbcd+4s^3bcd$, i. e. (dividing both ſides by $4sscd$) as $1\frac{1}{2}rb+sb$ to $rb+sb$. Which was to be found and demonſtrated.

SCHOLIUM.

IF any one had rather proceed herein by Indiviſibles, as in the preceding Prop. having divided the Axis EF (*Fig.*
196 140.) again into three equal parts, and aſſuming the Values of the Lines determined in the Hyperbola, *viz.* sb for the Abſciſſa EF, rb for the Tranſverſe Axis, $\frac{rcd}{b}$ for the *Latus Rectum*, $rscd+sscd$ for the Square of the Semiordinate FG, &c. then the loweſt and greateſt Circle of the Diameter HG will be, as $rscd+sscd$; and if you make, As the *La-*

tus Tranſv. rb to the *Latus Rectum* $\frac{rcd}{b}$, ſo is the ▭ D*ſ*E made of $rb + \frac{2}{3}sb$ into $\frac{2}{3}sb$ (i. e. $\frac{2}{3}rsbb + \frac{4}{9}ssbb$) to a fourth; there will come out $\frac{2}{3}rscd + \frac{4}{9}sscd$ for the ſecond Circle of the Diameter *hg*; and by the ſame Inference (as *rb* to $\frac{rcd}{b}$, ſo $rb + \frac{1}{3}sb$ into $\frac{1}{3}sb$ to a fourth) for the third Circle of the Diameter ℌ𝔊, $\frac{1}{3}rscd + \frac{1}{9}sscd$; ſo that theſe Indiviſibles (for which here, and in the precedent Prop. alſo, the partial circumſcribed Cylinders may be aſſumed) proceed in a double Series of Numbers, the firſt in a ſimple Arithmetical Progreſſion 3, 2, 1; the latter in a duplicate Arithmetical Progreſſion of Squares 9, 4, 1: and the ſame if you make further new Biſections, will neceſſarily happen *ad infinitum*, (the former Numbers, *e.g.* in the firſt Biſection will be $\frac{6}{6}\frac{5}{6}\frac{4}{6}\frac{3}{6}\frac{2}{6}\frac{1}{6}rscd$, the latter $\frac{36}{36}\frac{25}{36}\frac{16}{36}\frac{9}{36}\frac{4}{36}\frac{1}{36}sscd$, &c.) it is manifeſt from the Conſectaries of *Prop.* 21. *lib.* 1. that the whole Cylinder HK will in like manner be expreſſed by a double Series of Parts anſwering, in Number, to the Indiviſibles of the Conoid made by any Biſection; but in Magnitude, to the greateſt of them all: and the Sum of its firſt Series of Parts will be to the Sum of the firſt in the Conoid, both being infinite, as 2 to 1, or 3 to $1\frac{1}{2}rscd$, by *Conſect.* 9. of the ſaid *Prop.* 21. And the Sum of its latter to the Sum of the former in the Conoid, will be as 3 to $1sscd$; and ſo the whole Cylinder to the whole Conoid, as $3rscd + 3sscd$ to $1\frac{1}{2}rscd + sscd$; i. e. (dividing by *scd*) as $3r + 3s$ to $1\frac{1}{2}r + s$, i. e. (multiplying both ſides by *b*) as $3rb + 3sb$ to $1\frac{1}{2}rb + sb$; and conſequently the Cone (which is $\frac{1}{3}$ of the Cylinder) to the Conoid, as $rb + sb$ to $1\frac{1}{2}rb + sb$. Q. E. D.

p.76-80.

78.

CONSECTARY.

HENCE alſo appears the Proportion of the Hyperbolick Conoid to a Cylinder of the ſame Baſe and Altitude, which we did not expreſs in the *Prop.* which is as the Aggregate of the Axis and half the *Latus Tranſverſum* to triple the Aggregate of the ſaid Axis and *Latus Tranſverſum*.

◆

CHAP.

CHAP. IV.

Of Spiral Lines and Spiral Spaces.

PROPOSITION XVII.

THE * *firſt Spiral Space is ſubtriple of the firſt Circle,* i.e. *is to it as* 1 *to* 3.

* *Archim. Prop.* 24. *de Spiral.*

Demonſtration.

Having divided the Circumference of the Circle into (*Fig.*
p.196. 141. *n.* 1.) three equal parts by Lines drawn from the initial
Point, beginning from the firſt Line BA; the Line B*c* will be
148. as 1, BD as 2, BA as 3, by *Conſect.* 1. *Def.* 12. of this Book;
and conſequently the Sectors circumſcribed about the Spiral,
88. *viz.* CB*c* will be as 1, DB*d* as 4, AB*a* as 9, by *Prop.* 32.
lib. 1. In like manner, if you make new Biſections, the
Lines drawn from the Point B to the Spiral, will be 1, 2, 3, 4,
5, 6, but the circumſcribed Sectors 1, 4, 9, 16, 25, 36; and ſo
the circumſcribed partial Sectors *ad infinitum* will proceed in
an order of Squares, there being always as many Sectors in
the Circle equal to the greateſt of them. Therefore all the
Sectors that can be circumſcribed *ad infinitum* about the ſpi-
ral Space, *i.e.* the ſpiral Space itſelf (in which at laſt they
end) to ſo many equal to the greateſt, *i.e.* to the Circle, is
79. as 1 to 3, by *Conſ.* 10. *Prop.* 21. *lib.* 1. Q. E. D.

CONSECTARY I.

SINCE the firſt Circle is to the ſecond as 1 to 4 (*i.e.* as
147.88. 3 to 12) by *Def.* 12. of this, and *Prop.* 32. *lib.* 1. and the
firſt ſpiral Space to the firſt Circle, as 1 to 3, by the preſent
Prop. the ſame firſt ſpiral Space will be to the ſecond Circle
as 1 to 12; and to the third, by a like Inference, as 1 to 27;
the fourth, as 1 to 48, &c.

CONSECTARY II.

THE firſt ſpiral Line is equal to half the Circumference of the firſt Circle. For the Lines or Radii of the Sectors, and conſequently their Peripheries or Arches proceed in a ſimple Arithmetical Ratio, *viz.* as 1, 2, 3, 4, 5, 6, &c. while in the mean time the whole Periphery of the Circle contains ſo many Arches equal to the greateſt. Therefore the whole Periphery of the Circle is to an infinite Series of circumſcribed p.78.

cumſcribed Arches, *i.e.* to the ſpiral Line itſelf, as 2 to 1, by *Conſ.* 9. *Prop.* 21. *lib.* 1.

PROPOSITION XVIII.

THE whole ſpiral * *Space comprehended under the ſecond right Line* EA, *and the ſecond Spiral* EGIA (ſee Fig. 141. N. 2.) *is to the ſecond Circle, as* 7 *to* 12.

* *Archim. Prop.* 25.

Demonſtration.

For having firſt divided the Circumference of the Circle into three equal Parts, there will be drawn to the ſecond Spiral four Right Lines BE, BG, BI, and BA, being as 3, 4, 5, 6, and but only three Sectors circumſcribed, *viz.* GBg, IB*i*, and AB*a*, which proceed according to the Squares of the three latter Lines, *viz.* 16, 25, 36. ſo that the Sum is 77, while the Sum of three equal to the greateſt is 108; and ſo the one to the other (dividing both Sides by 9) as 12 to $8\frac{5}{9}$. Having moreover biſected the Arches and Parts of the Line BE, ſo that they be 6; then the ſecond BF will be 7, and ſo the other five will be 8, 9, 10, 11, 12; and the Sectors anſwering to them (excepting the firſt) 49, 64, 81, 100, 121, 144; ſo that their Sum ſhall be 559, while the Sum of ſix equal to the greateſt, *i.e.* the whole Circle, is 864: and ſo one to the other, (dividing both by 72) as 12 to $7\frac{55}{72}$. In another Biſection of the Arches and the Parts of the Line BE, ſo that they be 12, the ſecond thirteen, &c. to the thirteenth BA, which will be 24, the Sum of twelve Sectors will be found to be 4250; and the Sum of as many equal to the greateſt, will be 6912; and ſo the one to the other (dividing both ſides by 576) as 12 to $7\frac{218}{576}$, or $7\frac{109}{288}$. Therefore the Proportion will be

I. In the firſt caſe, 12 to $7 + 1 + \frac{1}{2} + \frac{1}{18}$ *viz.* $\frac{4}{72}$.
II. In the ſecond caſe 12 to $7 + \frac{1}{2} + \frac{1}{4} + \frac{1}{72}$ *viz.* $\frac{4}{288}$.
III. In the third caſe 12 to $7 + \frac{1}{4} + \frac{1}{8} + \frac{1}{288}$ &c.

The firſt and ſecond Fractions decreaſing by their halves, and the latter by $\frac{1}{4}$. Wherefore the Proportion of the ſecond Circle to the ſecond ſpiral Space will be as

$$\begin{array}{l} 12 \text{ to } 7 + 1 + \frac{1}{2} + \frac{1}{18} \\ \qquad - \frac{1}{2} - \frac{1}{4} - \frac{3}{72} \\ \qquad - \frac{1}{4} \text{ \&c.} - \frac{1}{8} \text{ \&c.} - \frac{3}{288} \text{ \&c.} \end{array}$$

By virtue of *Conſect.* 3. and 8. $= 0 = 0$
Prop. 21. *lib.* 1.
i.e. as 12 to 7, *Q.E.D.*

CONSECTARY I.

BEcauſe the ſecond Circle is to the firſt ſpiral Space, as 12 to 1, by *Conſect.* 1. of the preceding *Prop.* and to the ſecond ſpiral Space, as 12 to 7, by the preſent *Prop.* it will be to the ſecond Space without the firſt (*viz.* BCDEAIGE) as 12 to 6, *i.e.* as 2 to 1.

CONSECTARY II.

THerefore the ſecond Space ſeparately to the firſt, is as 6 to 1.

CONSECTARY III.

SINCE by triſecting both theſe Circles, *viz.* the firſt and ſecond, there ariſe ſix Lines, and as many Sectors, that is, the three Lines BC, BD, BE, *i.e.* 1, 2, 3, to which there anſwer three Arches in the ſame Progreſſion within the firſt Circle, and alſo as many equal to its greateſt; and theſe three other Lines BG, BI, BA, *i.e.* 4, 5, 6, to which there anſwer three Arcs in the ſame Progreſſion within the ſecond Circle, and likewiſe as many equal to its greateſt: Therefore the Sum of all the unequal Arches will be 21; but the Sum of the equal ones of both Circles, (each of which in the firſt, are equivalent to 3, in the ſecond to 6) will be 27. Wherefore the Sum of both the Peripheries to the Sum of all the circumſcribed Arches, will be as 27 to 21, *i.e.* (dividing both Sides by 9) as 3 to $2\frac{1}{3}$. Biſecting again the Arches of the Circles and the Parts of the Line BA, there will ariſe ſix circumſcribed unequal Arches within the firſt Circle, which are as 1, 2, 3, 4, 5, 6, and as many within the ſecond, which are as 7, 8, 9, 10, 11, 12; the Sum of all 12 is 78, while the Sum of as many equal to the greateſt is 108. Wherefore the one to the other, *i.e.* the Sum of both the Peripheries to twelve circumſcribed Arches taken together, is now as 108 to 78, *i.e.* (dividing both Sides by 36) as 3 to $2\frac{1}{6}$: and making another Biſection, the Proportion will be found to be as 3 to $2\frac{1}{12}$ &c. and hence at length may be evidently inferred, that the Sum of both the Peripheries will be to the Sum of all the circumſcribed Arches *ad infinitum*, *i.e.* to the whole Helix, as

$$3 \text{ to } 2 + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{12} \text{ \&c.} = 0\text{; that is, as } 3 \text{ to } 2.$$

Q.E.D.

CON-

CONSECTARY IV.

THerefore ſince the Periphery of the ſecond Circle is the double of the firſt, that alone will be equal to the whole Spiral.

CONSECTARY V.

THerefore if the Periphery of the ſecond Circle be 2, the Periphery of the firſt will be 1, and the firſt ſpiral Line $\frac{1}{2}$, by *Conſect.* 2. of the anteced. *Prop.* wherefore the ſecond Spiral alone will be 1 $\frac{1}{2}$; and ſo the Periphery of the ſecond Circle alone, will be to the ſecond Spiral alone, as 2 to 1 $\frac{1}{2}$, *i. e.* as 4 to 3; and to the firſt alone, as 4 to 1.

SCHOLIUM I.

BUT as *Conſect.* 4. may be alſo deduced after another way, *viz.* by comparing only the Arches of the ſecond Circle with the correſpondent circumſcribed Arcs, but conſidering them as taken twice, (becauſe that Circle is twice turned round, while the whole Helix or Spiral is deſcribed) and finding in the firſt Triſection the Proportion of double the ſecond Periphery to all the circumſcribed Arcs, as 12 to 7; and in the ſucceeding Biſection, as 12 to 6 $\frac{1}{2}$; in the ſecond Biſection, as 12 to 6 $\frac{1}{4}$, &c. and at length by inferring, that the ſecond Periphery is double of all the Arches circumſcribable about the whole Helix *ad infinitum*, that is, to the Helix itſelf,

$$\begin{aligned} \text{as } 12 \text{ to } 6 &+ 1\\ &-\tfrac{1}{2}\\ &-\tfrac{1}{4}\ \text{\&c.} = 0;\ \textit{i.e.}\ \text{as } 12 \text{ to } 6: \end{aligned}$$

and conſequently the ſecond Periphery alone will be to the whole Helix as 6 to 6: So the 5 *Conſect.* may be ſeparately had after the ſame manner, if inſtead of the firſt Triſection, you only biſect, (*vid. Fig.* 141. *N.* 3.) for thus in the firſt Biſection the Arches circumſcribed about the ſecond ſpiral Line will be ſeparately two Semi-Circles D*d*, 3, and A*a*, 4, (for as the Line BC is 1, BE 2, BD 3, BA 4; ſo the Arch deſcribed by the Radius BD is 3, and that deſcribed by the Radius BA = 4) and their Sum is 7; while the Sum of two equal to the greateſt is 8. In the ſecond Biſection (ſince BE is 4) BF and its Arch is 5, the Arch BD 6, the Arch BG 7, the Arch BA 8, and the Sum is 26; while the Sum of ſo many Quadrants equal to the greateſt is 32. So in the third Biſection, the Sum of eight Octants circumſcribed about the ſecond Helix will be found to be 100, and the Sum of ſo many = to the greateſt is 128, &c. Wherefore the Periphery of the ſecond Circle in the firſt Caſe, will be to the Sum of the Arches circumſcribed about the ſecond Helix, as 4 to $3 + \frac{1}{2}$; in the ſecond, as 4 to $3 + \frac{1}{4}$; in the third, as 4 to $3 + \frac{1}{8}$ &c. and ſo to all the Arches circumſcribable *in infinitum*, *i. e.* to the ſecond Helix itſelf, as

$$4 \text{ to } 3 + \tfrac{1}{2} \\ - \tfrac{1}{4} \\ - \tfrac{1}{8} \text{ \&c.} = 0; \textit{ i. e.} \text{ as 4 to 3.}$$

Q. E. D.

By the ſame Method you may eaſily find the Proportion of the third Circle to the third ſpiral Space, and of that Periphery either to the whole Spiral, or ſeparately to the third, as will be evident to any one who tries.

I. *For the third ſpiral Space.*

(*Fig.* 142.)

BC	1	BF	4	BI	7	49
BD	2	BG	5	BK	8	54
BE	3	BH	6	BA	9	81

are the three firſt Sectors circumſcribed about the Parts of the third Helix. The Sum of theſe three Sectors is 194; and the Sum of ſo many equal to the greateſt 243. Therefore the firſt Proportion of the one Sum to the other, will be, as 243 to 194, *i.e.* (dividing both Sides by 9) as 27 to $21\frac{1}{9}$.

In the firſt Biſection there will be ſeven Lines:

BH 12,	BL	13	169
	BI	14	196
	BM	15	225
	BK	16	256
	BN	17	289
	BA	18	324
	Sum		1459;

Sectors circumſcribed about the Parts of the third Helix.

while in the mean time the Sum of as many equal to the greateſt is 944; and ſo the ſecond Proportion is as 1944 to 1459; *i.e.* (dividing both Sides by 72) as 27 to $20\frac{19}{72}$.

In the ſecond Biſection there will be thirteen Lines, *viz.* BH 24, the reſt 25, 26, &c. but the Sum of the Sectors, *i. e.* of the Square Numbers anſwering to the twelve latter, will be found to be 11306; while in the mean time the Sum of as many equal to the greateſt will be 15552; ſo that you will

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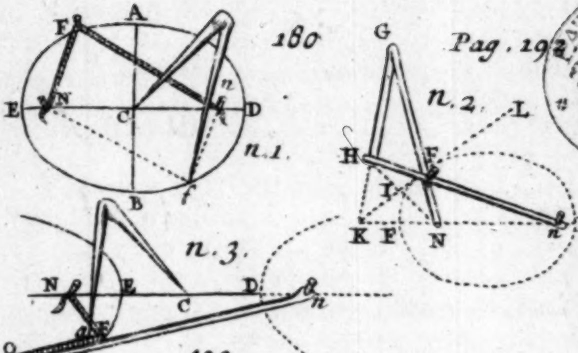

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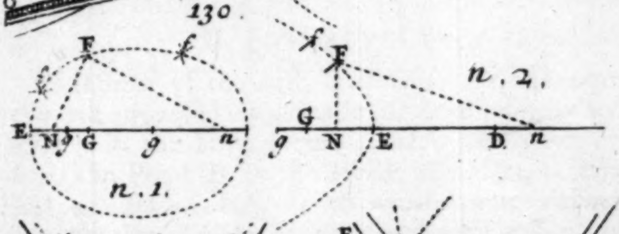

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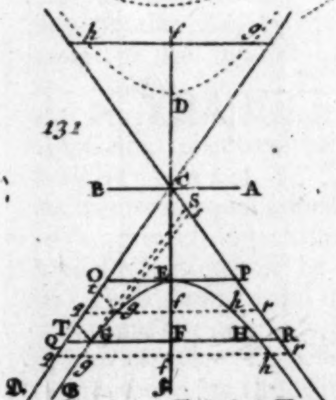

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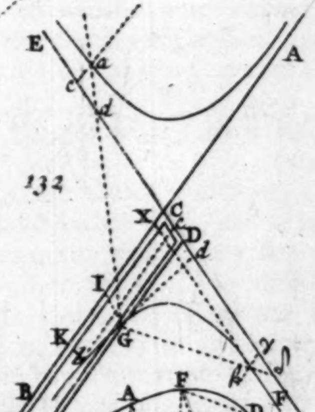

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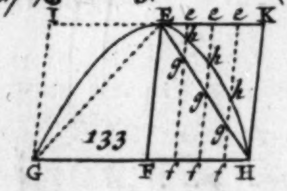

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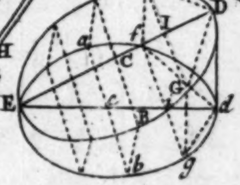

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have the third Proportion of this Sum to the other, as 15552 to 11306, *i.e.* (dividing both ſides by 576) as 27 to $19\frac{362}{576}$.

Therefore the I^st^ Proportion will be as 27 to $19 + 2 + \frac{5}{9}$, *i.e.* to $19 + 2 + \frac{1}{2} + \frac{1}{18}$.

The II^d^ . . . . as 27 to $19 + 1 + \frac{19}{72}$, *i.e.* to $19 + 1 + \frac{1}{4} + \frac{1}{72}$.

The III^d^ . . . . as 27 to $19 + \frac{362}{576}$, *i.e.* to $19 + \frac{1}{2} + \frac{1}{8} + \frac{1}{288}$.

Therefore the Proportion of the third Circle to the third ſpiral Space will be

$$\left.\begin{array}{llll} \text{as } 27 \text{ to } 19 & + 2 & + \frac{1}{2} & + \frac{1}{18} \\ & - 1 & - \frac{1}{4} & - \frac{1}{72} \\ & - \frac{1}{2}, \&c. & - \frac{1}{8}, \&c. & - \frac{1}{288}, \&c. \\ & = 0. & = 0. & = 0. \end{array}\right\} \begin{array}{l} \text{that is, as} \\ 27 \text{ to } 19. \\ \text{Q. E. D.} \end{array}$$

II. *For the third Spiral Line.*

If inſtead of the firſt Triſection (as leſs commodious for the end propoſed) you make uſe here alſo, as before, of Biſection in the ſame Figure; there will come out ſix Lines from the Point B to the Helix, *viz.* B*m*, 1, BE, 2, B*n*, 3, BH, 4, B*o*, 5, BA, 6; to which there anſwer as many ſemicircular Arches in the ſame Progreſſion, and as many equal to the greateſt four times taken; ſo that the Sum of the unequal ones is 21, and of the equal ones 24, and ſo the Proportion of three Peripheries together, to all the circumſcrib'd ones together, will be as 24 to 21, (and dividing both by 6) as 4 to $3\frac{1}{2}$. In the ſecond Biſection, the Sum of the twelve unequal Lines and Arches is 78, and as many equal to the greateſt eight times taken will be 96; ſo that the ſecond Proportion will be 96 to 78, *i.e.* (dividing both ſides by 24) as 4 to $3\frac{1}{4}$. In the third Biſection, the Proportion will come out as 384 to 300, *i.e.* (dividing both ſides by 96) as 4 to $3\frac{1}{8}$, &c. Therefore the Proportion of the three Circles together, to the whole Helix, will be as 4 to

$$3 \begin{array}{l} + \frac{1}{2} \\ - \frac{1}{4} \\ - \frac{1}{8}, \&c. \end{array}$$

$= 0$, *i.e.* as 4 to 3, or 12 to 9. Q. E. D.

CONSECTARY VI.

NOW, if the Periphery of the firſt Circle be made 2, that of the ſecond will be 4; of the third, 6; and conſequently the Sum will be 12: it will be manifeſt, that the third Periphery ſeparately will be to the whole Helix as 6 to 9, *i.e.* as 2 to 3.

CONSECTARY VII.

AND becauſe the ſecond Periphery (which is 4) is equal to the firſt and ſecond Helix together, by the above *Conſect.* 4. the remaining third Spiral will be 5, and ſo the Proportion of the third Periphery to it as 6 to 5.

CONSECTARY VIII.

WHerefore the Proportions of each of the Peripheries to their correſpondent Spirals, will be in a Progreſſion of ordinal Numbers, *viz.* ſo that the latter of every two will denote the Periphery of a Circle, and the former an inſcribed Spiral; and conſequently the ſpiral Lines will be in an arithmetical Progreſſion of the odd Numbers, and the Peripheries of the Circles in a Progreſſion of the even ones.

1 . . . . The firſt Spiral,
2 . . . . The firſt Periphery,
3 . . . . The ſecond Spiral,
4 . . . . The ſecond Periphery,
5 . . . . The third Spiral,
6, *&c.* . The third Periphery, *&c.*

SCHOLIUM II.

THE ſeventh Conſectary may alſo be eaſily deduced ſeparately this way: In the firſt Biſection, the Line BA and its Periphery is 6; the Line B*o*, and its Periphery, 5; and the Sum of the circumſcribed Peripheries 11; the Sum of as many equal to the greateſt 12. Therefore the Periphery of the third Circle will be to the two circumſcribed ones, as 12 to 11, *i. e.* as 6 to $5\frac{1}{2}$. In the ſecond Biſection, the four circumſcribed Quadrants will be 12, 11, 10, 9, and their Sum 42; and the Sum of four equal to the greateſt, *i. e.* the Periphery of the third Circle is 48. Therefore the Proportion is now as 48 to 42, *i. e.* (dividing both ſides by 8) as 6 to $5\frac{1}{4}$. Thus you will have the third Proportion, as 192 to 164, *i. e.* (dividing both ſides by 32) as 6 to $5\frac{1}{8}$. Wherefore the Proportion of the third Periphery to the third Helix, or Spiral, is as 6 to

$$\begin{array}{l} 5 + \frac{1}{2} \\ \quad - \frac{1}{4} \\ \quad - \frac{1}{8}, \text{\&c.} \\ \quad = 0. \end{array}$$

i. e. as 6 to 5. Q. E. D.

CONSECTARY IX.

AS *Conſect.* 8. ſupplies us with a Rule to determine the Proportion of every Spiral of every Order to the Periphery of the correſpondent Circle, *viz.* if the Number of the Order be doubled for the Periphery of the Circle, and the next antecedent odd Number be taken for the ſpiral Line; ſo what we have hitherto demonſtrated ſupplies alſo another Rule, to define the Proportion of the ſpiral Space in any Order to its Circle. For ſince the Circles are in a Progreſſion of Squares 1, 4, 9, 16, *&c.* and the firſt Circle is to the firſt Space, as 3 to 1, (*i. e.* as 1 to $\frac{1}{3}$) by *Prop.* 17. and the ſecond to the ſecond, as 12 to 7 (*i. e.* as 4 to $2\frac{1}{3}$) by *Prop.* 18. the third to the third, as 27 to 19 (*i. e.* as 9 to $6\frac{1}{3}$) by *Schol.* 1. of this; contemplating both theſe Series one by another, *viz.*

This of the Circles, —— 1, 4, 9.
And this of the ſpiral Spaces, $\frac{1}{3}$, $2\frac{1}{3}$, $6\frac{1}{3}$.

We ſee the Numbers expreſſing the Spaces are produced, if from the ſquare Numbers of the Circles you ſubſtract their Roots, and add $\frac{1}{3}$ to the Remainder. Wherefore if, *e. g.* we were to determine the Proportion of the fourth Circle to the fourth ſpiral Space, the Square of 4, *viz.* 16 would give the Circle; and hence, ſubſtracting the Root 4, there would remain 12, and adding $\frac{1}{3}$, you would have the fourth ſpiral Space $12\frac{1}{3}$; and in like manner the ſpiral Space $20\frac{1}{3}$ would anſwer to the Circle 25, *&c.* And that this is certain, is hence evident, if we multiply theſe Numbers 16 and $12\frac{1}{3}$, alſo 25 and $20\frac{1}{3}$ by 3, ſo as to get thoſe Proportions in the whole Numbers 48, 37, 75, and 61; theſe are thoſe very Numbers *Archimedes* had hinted at in the *Coroll.* of *Prop.* 25.

CONSECTARY X.

NAY, what we have now ſaid, is that very *Coroll.* comprehending alſo that 25th *Propoſition*, viz. *That a ſpiral Space of any Order is to its correſpondent Circle, as the Rectangle of the Semidiameters of this and the preceding Circle, together with a third Part of the Square of the Difference between both Semidiameters to the Square of the Semidiameter of the greateſt Circle.* For if, *e. g.* the Proportion of the third ſpiral Space to the third Circle be required; ſince the Semidiameter of this third Circle is as 3, and the Semidiameter of the ſecond precedent one is 2, and ſo the Difference 1; the Rectangle of 2 into 3, *i.e.* 6, together with $\frac{1}{3}$ of the Square of the Difference, will define the third ſpiral Space $6\frac{1}{3}$; ſince the third Circle may be defined by the Square of the Semidiameter of the greater, *viz.* by 9, and ſo in the reſt; as the Numbers we have found ſhew, or that may be found according to the given Rule, which may be here ſeen in the following Table.

Orders.	I	II	III	IV	V	VI	VII	VIII	IX	X
Circles.	1	4	9	16	25	36	49	64	81	100
The whole Spaces, the preced. ones being included.	$1\frac{1}{3}$	$2\frac{1}{3}$	$6\frac{1}{3}$	$12\frac{1}{3}$	$20\frac{1}{3}$	$30\frac{1}{3}$	$42\frac{1}{3}$	$56\frac{1}{3}$	$72\frac{1}{3}$	$90\frac{1}{3}$
Separate Spaces, the preced. ones being excluded.	$\frac{1}{3}$	2	4	6	8	10	12	14	16	18

CONSECTARY XI.

FRom which Table * it is obvious to ſight, that the ſecond Space excluding the firſt, is ſextuple of the firſt, as we have already deduced in *Conſect.* 2 *Prop.* 18. the third ſeparate Space is double of the ſecond; the fourth triple of the ſame ſecond; the fifth quadruple, and ſo onwards.

* *Archimed. Prop.* 27.

SCHOLIUM III.

AND thus much for Spirals, we having not only demonſtrated the chief Theorems of *Archimedes* concerning ſpiral Spaces, but alſo the chief of ſpiral Lines, (whereof *Archimedes* has left nothing.) If any ſhould have a mind to carry on our Method further, he may, after the ſame way, eaſily demonſtrate what remains in *Archimedes*, and what Dr. *Wallis*, in his Arithmetick of Infinites, from *Prop.* 5, to the 38th, and others have done on this Argument.

CHAP.

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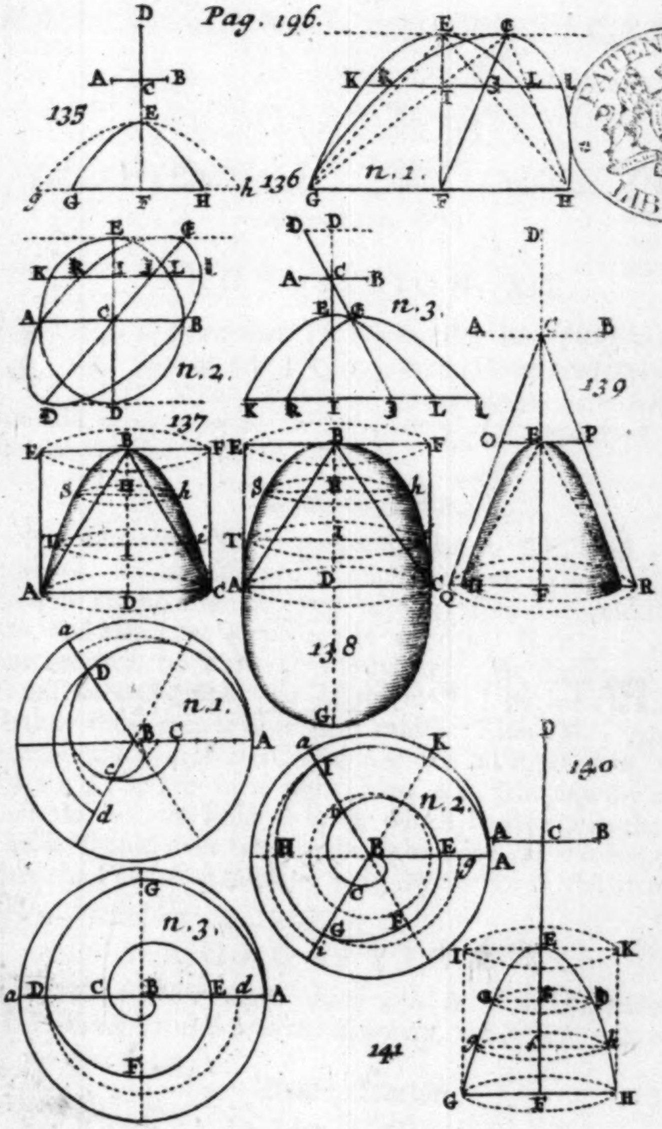

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CHAP. V.

Of the Conchoid, Ciſſoid, Cycloid, Quadratrix, &c.

PROPOSITION XIX.

THE firſt Conchoid of Nicomedes B*bb* (Fig. 110.) *on both ſides of the perpendicular c*D*b approaches nearer always to the Directrix or horizontal Line* AE, *and yet will never coincide with it, altho' it be conceived to be produced on both ſides* ad infinitum.

Demonſtration.

For ſince only D*b* is perpendicular to AE, and all the others *ab*, *ab*, *ab*, &c. are ſo much the more inclined to it, by how much the more remote they are from the middle one D*b*, and all in the mean while are equal both to it and to one another, by *Def.* 13. it is evident, that the Points *b* and B will come ſo much the nearer to AE, by how much the farther they recede from the middle Line D*b*. And yet becauſe the Lines BAC and *bac* are all right ones, whoſe Points A, *a*, are in a right Line AE, it is equally as impoſſible, that the Point *b* or B, which is always in the Conchoid, ſhould ever touch this right Line, as it is impoſſible that the Point C ſhould be in it, by virtue of the aforecited *Def.* Q. E. D.

PROPOSITION XX.

YET no other right Line can be drawn between the Directrix AE *and the Conchoid, but what will cut it, if produced.*

Demonſtration.

For if ſuch a right Line as GH be ſuppoſed parallel to AE, and you make, as DI to IC, ſo is D*b* to a fourth, which will be greater than IC, ſince D*b* is greater than DI; and conſequently, if making that fourth Proportional a Semidiameter you draw a circular Arch from C, it will neceſſarily cut the Line GH, *e.g.* in G. Drawing therefore C*a*G, you will have DI to IC, as *a*G to GC, *i. e.* to that fourth Proportional before found, by virtue of *Prop.* 34. *lib.* 1. but as DI to IC, ſo was alſo D*b* to the ſaid fourth Proportional by Conſtruct. Therefore *a*G and D*b*, which have both the ſame Proportion to the ſame Quantity, are equal; and conſequently the Point G is in the Conchoid, by virtue of *Def.* 13. and therefore the right Line GH being produced, will cut that produced alſo, on both ſides, by the ſame reaſon. Much more will it cut it on either ſide, if it be not parallel to the Directrix AE, as is very obvious. Therefore no right Line can be drawn between the Conchoid, *&c.* Q. E. D.

CONSECTARY.

HENCE, beſides other Problems, that may be very eaſily ſolved by this Curve, you may ſolve this; *viz. Having any rectilinear Angle* ABC (*Fig.* 143.) *and a Point without it given, from that Point to draw a right Line* DEF, *ſo that that part of it* EF, *which is intercepted between the Legs of the Angle, ſhall be equal to a given Line* Z. For if you draw the Perpendicular DGH from the given Point D, thro' the neareſt Leg BC of the Angle, and make GH equal to the given Line Z; and with the Center C, at the Interval GH, deſcribe the Conchoid IHK, which will be neceſſarily cut by the other Leg of the Angle, by virtue of the preſent *Prop. e.g.* in F; then the Line DF being drawn, will give the intercepted Part EF = GH, by the nature of the Conchoid, and conſequently = to the given Line Z.

SCHOLIUM.

BY means of this Conſectary, *Nicomedes* ſolves that noble Problem of finding two mean Proportionals, after this way, which we will here ſhew from *Eutocius*, but drawn into a Compendium, and ſomewhat changed as to the Order. Let two given Lines AB and BC (*Fig.* 144.) between which you are to find two mean Proportionals, be joined together at right Angles, and divide both into two Parts in D and E; and having compleated the Rectangle ABCL, from L thro' D, draw LG to BC prolonged, that ſo GB may be = AL or BC: Having let fall a Perpendicular from E, from C as a Center, with the Interval CF = AD, cut off the Part EF, and having drawn FG, make CH parallel to it; and laſtly, thro' the Legs of the Angle KCH draw the right Line FHK, ſo that the Part HK ſhall be equal to the Line CF, by the preceding Conſect. and alſo draw the right Line KM from K thro' L to the Line BA continued: All which being done, CK and AM will be two mean Proportionals between AB and BC; which, after our way, we thus demonſtrate. By reaſon of the Similitude of the Triangles MAL and LCK,

MA is to LC or AB, as AL or BC to CK

b . . . . eb . . . . c . . . . ec

and moreover,

as MA to AD, ſo is GC to CK, *i. e.* FH to HK

b . . . . $\frac{1}{2}eb$ . . . . $2c$ . . . ec, becauſe GF and CH are parallel, by *Conſect.* 4. *Prop.* 34. *lib.* 1. Therefore ſince HK is $= \text{AD} = \frac{1}{2}eb$, FH will be $= \text{MA} = b$, and conſequently $\text{MD} = \text{FK}$, *viz.* Each $b + \frac{1}{2}eb$, and the Square of each $= bb + ebb + \frac{1}{4}eebb = \square$ of EF $+ \square$ EK, by virtue of the *Pythag. Theor.* Now if to theſe equal Quantities you add the equal Squares of DX and EC $= \frac{1}{4}cc$, their Sum, *viz.* $\square$ of MD $+ \square$ of DX, *i. e.* $\square$ of MX will be $bb + ebb + \frac{1}{4}eebb + \frac{1}{4}cc$, equal to the Sum of theſe, *viz.* $\square$ of EF $+ \square$ of EC, *i. e.* $\square$ of CF (by the *Pythag. Theor.* or EX by Conſtruct.) $+ \square$ of EK, $\frac{1}{4}eebb + \frac{1}{4}cc + ecc + eecc = \square$ KX; whence theſe two things now follow: 1. That the Lines MX and KX are equal. 2. If from thoſe equal Sums you take away the common Quantities $\frac{1}{4}eebb + \frac{1}{4}cc$, the Remainders will be equal, *viz.* $bb + ebb = ecc + eecc$; and (ſince the Part taken away, *viz.* bb is manifeſtly to the other Part taken away, *viz.* ecc, as the Remainder ebb to the Remainder $eccc$, and the whole together with the Parts taken away, and the Remainders are in the ſame Proportion, by *Prop.* 26. *lib.* 1.) ſeparately alſo bb will $= ecc$ and $ebb = eecc$. But from the latter Equation, it follows, that

as eb to ec, ſo is ec to b, by virtue of *Prop.* 19. *lib.* 1.

AB to CK, ſo CK to MA;

and by the ſame reaſon it follows, from the former Equation, that

ec to b, ſo b to c

CK to MA, ſo MA to BC, *i. e.* CK and MA are the two mean Proportionals between AB and BC. Q. E. D.

From which Deduction you have alſo the Foundation of that mechanical way, which *Hiero Alexandrinus* makes uſe of in *Eutocius, lib.* 2. *of the Sphere and Cylinder*, and which *Swenterus* has put into his practical Geometry, *lib.* 1. *Tract.* 1. *Prop.* 23. *viz.* having joined the given right Lines AB and BC in the form of a Rectangle, (*Fig.* 145.) and continued them out, he ſo long moves a Ruler backwards and forwards in L, as about a Center, till XK and XM, by help of a Pair of Compaſſes, are found equal. To which, another way of *Philo*'s is not unlike, and flows from the ſame Fountain, wherein, having made a Semicircle on AC, the moveable Ruler in L is ſo long moved backwards and forwards, until LM and NK are found equal: Which ſeems to *Eutocius* to be more accommodated to Practice, and eaſier to be perform'd, by help of a Ruler divided into ſmall equal Parts.

PROPOSITION XXI.

If from any Point, as G *of the Diameter* CD *in the generating Circle* (Fig. 111. N. 1.) *you draw a Perpendicular* GE *thro' the Ciſſoid of* Diocles, *the Lines* CG, GE, GD, GH *will be continual Proportionals.*

Demonſtration.

For ſince GE and IF, as right Sines, and alſo GD and IC, as verſed Sines of equal Arches ED, FC, by the Hypoth. are equal; it will be as ID to IF, (*i.e.* CG to GE) ſo is IF to IC (*i.e.* GE to GD) *per N.* 3. *Schol.* 2. *Prop.* 34. *lib.* 1. But GD is to GH, as ID to IF (*i.e.* as GE to GD) by the forecited *Prop.* 34. *lib.* 1. Therefore CG to GE, GE to GD, and GD to GH, are all in the ſame continual Proportion. Q. E. D.

CONSECTARY.

HENCE it was eaſy for *Diocles* to find two mean Proportionals X and Y between two given right Lines V and Z; (*Fig.* 146.) for having firſt deſcribed his Curve DHB, he made as V to Z, ſo is CL to LK; and having drawn CKH to the Curve, and thro' H the Perpendicular GE, he had between CG and GH two mean Proportionals GE and GD, by virtue of the preſent *Prop.* when in the mean while CG the firſt would be to GH the laſt, as CL to LK, *i.e.* as the firſt V to the laſt Z, by virtue of the Conſtruct. Therefore nothing remain'd but to make, 1. As CG to GE, ſo V to X; and, laſtly, as GE to GD, ſo X to Y.

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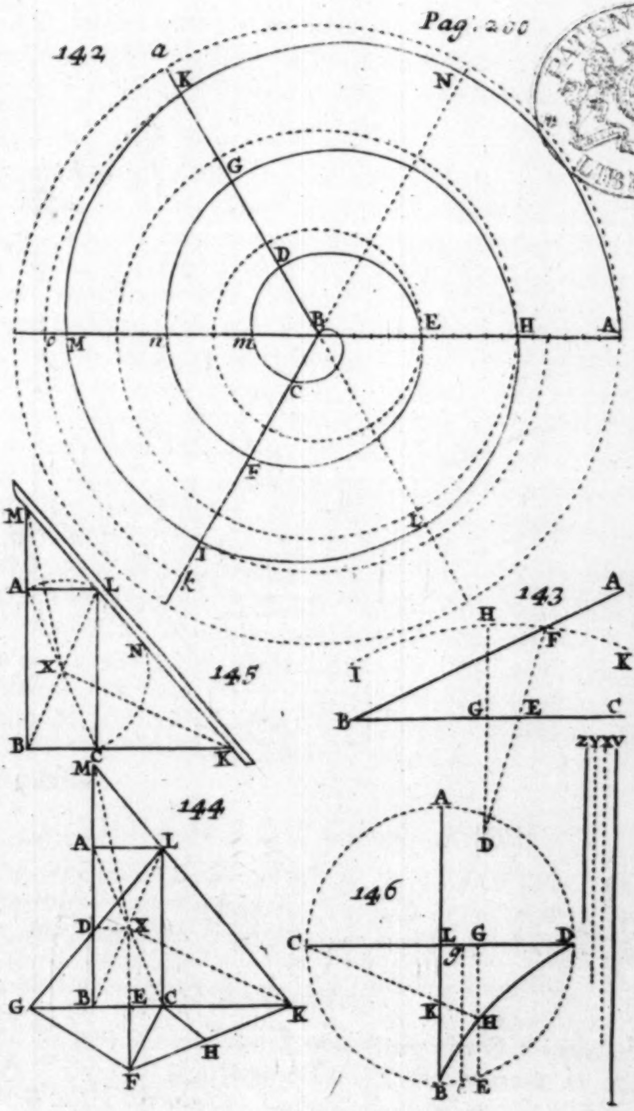

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SCHOLIUM.

IT may not be amiſs to mention here another way of finding two mean Proportionals between any two given Lines, by the help of two Parabola's, which *Menechmus* formerly made uſe of, *viz.* by joining at right Angles the given Lines AB and BC (*Fig.* 147.) and prolonging them as occaſion ſhall require thro' E and D; and then deſcribing a Parabola about BE as its Axis, ſo made that BC ſhall be its *Latus Rectum*; and in like manner deſcribing another Parabola about BD as its Axis, that ſhall have AB for its *Latus Rectum*, and that ſhall cut the former in F. Which being done, the Semi ordinate FE (or BD which is equal to it) being drawn to the Point of Interſection F, and the Abſciſſa BE will be the two mean Proportionals ſought. For by virtue of the fourth *Conſectary* of *Prop.* 1. of this Book, DF or BE is a mean Proportional between AB and BD; and in like manner EF or BD is a mean Proportional between BE and BC, and conſequently as AB to BE, ſo is BE to BD, and as BE to BD, ſo is BD to BC. Q. E. F.

To this way of *Menechmus*, that of *Des Cartes* is not unlike, which he gives us, *p. m.* 91. except only that inſtead of two Parabola's, he makes uſe only of one, and a Circle in room of the other: In imitation of whom, *Renatus Franciſcus Sluſius* has ſince ſhewn infinite Methods of doing the ſame thing by help of a Circle, and either Ellipſes or Hyperbola's, in his ingenious Treatiſe which he thence names his *Meſolabium*.

PROPOSITION XXII.

ANY Semi-ordinate (BF *or bf*) *of the Cycloid* (Fig. 148.) *is equal to its correſponding Sine* BD *or bd in the generating Circle, together with the Arch* AD *or* Ad *of that Sine.*

Demonſtration.

For the Motion of the Point A deſcribing the Semi-cycloid AFE, by *Def.* 11. is compounded of the Motion of the Circle (or Wheel) about the Center B thro' the Semicircle ADC, and of the Motion of the Center along the right Line B equal to CE, and conſequently = to the Semicircle itſelf, or Motion of the Circle. Therefore as the Point A, moving to E by the Motion of the Circle, moved or was carried from the Diameter AC thro' the whole Semicircle ADC till it came to AC again, and by the Motion of the Center paſſes thro' the whole Space BG or GE, which is equal to the ſemicircular Arch; ſo the ſaid Point A, when come to F, will have deſcrib'd the Quadrant AD, by moving from the Diameter AC by the Quantity of the Sine BD; and moreover, by the Motion of the Center (which is equal to the Motion of the Circle) moves from AC by the Space DF: And the Semi-ordinate BF will be equal to the Arch AD, and to its Sine BD taken together; and in like manner the Semi-ordinate *bf* will be equal to the Arch A*d*, and its Sine *bd*, &c. Q. E. D.

CONSECTARY I.

HEnce, by help of the Cycloid, you may eaſily aſſign a right Line equal to the Semi-periphery of the whole Circle, or any given Arch AD or A*d*, *viz.* CE doubled, or taken twice, for the whole Circumference; and ſingle, for the Semi-periphery, or half Circumference; DF, for the Quadrant A*d*; *df*, for the Arch A*d*, &c.

CONSECTARY II.

WHerefore the Quadrature of the Circle may be geometrically obtain'd, according to *Conſect.* 2. of *Def.* 15. *lib.* 1.

CONSECTARY III.

IF you take B*e*, *be*, the doubles of the Sines BD, *bd*, &c. ſo that all the Indiviſibles *bd* taken together, may be to all the Indiviſibles *be* taken together, as BD to B*e*, a Curve deſcribed thro' the Points *e* will be an Ellipſis, by *Prop.* 11. and the Curvilinear Space ADC*e*A will be equal to the Semicircle ACDA.

CONSECTARY IV.

AND ſince DF (*i. e.* D*e*+*e*F) is equal to the Quadrant DA, by virtue of the preſent *Prop.* BD+FG will be alſo equal to the Quadrant (becauſe the whole BG or CE is = to a Semicircle) and conſequently *e*F and FG will be equal; in like manner, ſince *df*, both above and below, is equal to the Arch *d*A: below, *bd*+*fg* will be = to the remaining Arch *d*C: and above, *bd*+*ef* (*i. e.* *df*) will be = to the Arch *d*A. Therefore *ef* above, and *fg* below are equal, and (ſince the ſame may be ſhewn of all the Indiviſibles of the ſame ſort throughout) the trilinear Figure FGE will be equal to the trilinear Figure *e*FA.

PRO-

PROPOSITION XXIII.

THE cycloidal Space is triple of the generating Circle, or the ſemi-cycloidal Space AECA *is triple of the Semicircle* ADCA.

Demonſtration.

Since the Parallelogram BCEG is equal to the whole Circle, by *Conſect.* 2. of *Def.* 15. *lib.* 1. *i.e.* to the Semi-ellipſe A*e*CA, by the preſent Conſtruction; the Trapezium C*e*GE will be equal to a Quadrant of the Ellipſis or Semicircle. But the trilinear Space FEG is = to the trilinear Space FA*e*, by the fourth *Conſect.* preced. therefore alſo the trilinear Space A*e*CEA is equal to the Semi-circle. Therefore the whole cycloidal Space is equal to the three Semi-circles. Q. E. D.

Or thus;

Since the whole Parallelogram AE is equal to the two Circles, and the Semi-Ellipſis A*e*CA to one; the remaining Space A*e*CEC to one Circle, and its half A*e*GC to a Semi-circle: but the trilinear Space A*e*F, equal to the trilinear Space FGE, by *Conſect.* 4. preced. Therefore the one being ſubſtituted in the other's place, the trilinear Space AFEC will be equal to the Semi-circle: Therefore the Remainder of the Parallelogram, *i.e.* the cycloidal Space AFECA will be equal to the three Semi-circles. Q. E. D.

SCHOLIUM.

TO theſe ſhort Demonſtrations, which we confeſs we owe for the moſt part to *Hon. Faber*, we will ſubjoin another ſomewhat more prolix; but yet not unpleaſant, which we find in *Carolus Renaldinus*, *lib.* 1. *de Reſol. & Compoſ. Math. p.*299. But here we will give it the Reader more plain, and free from all Scruples, and likewiſe much eaſier. It is perform'd in theſe Inferences, 1. That the right-lined Parallelogram A*ba*B (*Fig.* 149. *N.*1.) is equal to the curvilinear Space A*bda*BDA. 2. That as that is divided into two equal Parts by its right-lined Diagonal A*a*, ſo likewiſe is this by the Semi-cycloid A*aa*; ſo that the right-lined Triangle A*a*B is equal to the curvilinear Space A*aa*BDA. 3. Therefore the one, as well as the other, is equal to the generating Circle; and conſequently, 4. If to this curvilinear Space there be added the Semi-circle ADBA, the ſemi-cycloidal Space A*aa*BA will be equal to three Semi-circles. The firſt Inſe-

Inference is evident; for if from the right-lined Parallelogram you take the Semi-circle ADBA on the one ſide, and on the other, add the Semi-circle *abda*, there will ariſe the curvilinear Parallelogram. The third Inference is evident from *Conſect.* 2. *Def.* 5. *lib.* 1. Becauſe the Line B*a* is equal to the Semi-periphery, which, multiplied by the Semi-diameter BC, gives the Area of the Circle. The fourth is ſelf-evident; and ſo there remains only the ſecond to be demonſtrated, *viz.* That the curvilinear Parallelogram is divided into two equal Parts by the Cycloid, *i. e.* that the external trilinear Figure A*adb*A is equal to the internal one A*aa*BDA; which may be thus ſhewn: Having divided the Baſe B*a* into three equal Parts, and drawn thro' them three Semi-circles, and moreover the tranſverſe right Lines D*d* and E*e* thro' the Interſections of the Semi-circles and the Cycloid, it is certain, from the Geneſis of the Cycloid, by virtue of the *Conſ.* of *Def.* 11. that as the right Line *a*1 is a third Part of the Whole *a*B, ſo the Arch 1*a* is a third Part of the generating Periphery; and by the ſame reaſon, the Arch 2*a* two thirds, and ſo the remaining Arch *a*II alſo $\frac{1}{3}$; inſomuch that the firſt Arch 1*a*, and the laſt *a*II, and conſequently their right Sines *af*, *ag*, and likewiſe their verſed ones *f*1, *g*II, are equal; and ſo the curvilined partial Parallelograms, both above and below, all upon equal Baſes, and of the ſame Height (*viz.* the two ſhaded ones *ae*21, and *da* II. I) are equal among themſelves, and ſo likewiſe the two pricked or pointed ones D*a* 2B, and *ae b*1. Wherefore if now the Baſe B*a* (*N.* 2.) be conceived to be divided into ſix equal Parts, and having drawn Semi-circles, and tranſverſe Lines thro' their Interſections with the Cycloid, the Arches

$$\left.\begin{matrix}1a\text{ and}\\ \mathrm{V}a\end{matrix}\right\}\text{will be }\tfrac{1}{6},\ \left.\begin{matrix}2a\text{ and}\\ \mathrm{IV}a\end{matrix}\right\}\tfrac{1}{3},\ \left.\begin{matrix}2a\text{ and}\\ \mathrm{III}a\end{matrix}\right\}\tfrac{1}{2}\text{ of the Semi-circle;}$$

and ſo the verſed Sines of each, *i. e.* the Altitudes of the correſponding Parallelograms will be equal, and conſequently the Parallelograms of the internal and external trilinear Space that are dotted alike, or mark'd, will be equal to each other. Now ſince this Inſcription of curvilinear Parallelograms, always reſpectively equal both in Number and Magnitude, may be continued in both the trilinear Figures *ad infinitum*, it will evidently follow, that the trilinear Figures themſelves, wherein theſe inſcrib'd Figures infinite in number are always equal, will be likewiſe equal to one another.

PRO-

PROPOSITION XXIV.

THE Baſe of the Quadratrix AE (Fig. 150.) *the Semidiameter of the generating Qudrant* AD, *and the Quadrant itſelf* BD, *are in continual Proportion.*

Demonſtration.

For the Quadrant DB is to the Radius DA, as the Arch IB to the Perpendicular H*e*, by *Conſect.* 1. *Def.* 16. and I*b* is to H*e*, as A*b* to A*e*, by *Prop.* 34. *lib.* 1. But the Arch IB being conceived leſs and leſs *ad infinitum*, will at length coincide with I*b*, as ending in the ſame moment in the Point B, wherein H*e* will end in the Point E, and ſo A*e* will end in AE, and A*b* in AB. Therefore at length DB will be to DA, as IB (*i. e.* I*b*) to H*e*, *i. e.* as A*b* to A*e*, *i. e.* as AB (or DA) to AE. Q. E. D.

SCHOLIUM I.

C*LAVIUS*, about the end of the ſixth Book of *Euclid*, and others, demonſtrate this indirectly, by a Deduction *ad abſurdum*, or concluding the contrary much after this manner: If DA or AB be not to AE, as DB to DA, ſuppoſe it to be ſo to a greater A*f*, or a leſs A*e*. In the firſt caſe therefore, becauſe AB is to A*f*, as DB to DA, *per Hypoth. i. e.* as K*f* to A*f*; the Quadrant K*f*, and the Radius AB or DA will be equal. But as BD is to IB, ſo is K*f* to H*f*, by reaſon of the Similitude of the Arches; and as BD to IB, ſo is alſo DA (=K*f*) to the Sine H*e*, by *Conſect.* 1. *Def.* 16. Therefore the Sine H*e*, and the Arch H*f* (to which the ſame K*f* bears the ſame Proportion) will be equal; which is abſurd. In the latter caſe, becauſe AB would be to A*e*, as DB to DA by the Hypoth. *i. e.* as L*e* to A*e*, the Quadrant L*e*, and the Radius AB, or DA, would be again equal. But as BD is to IB, ſo is L*e* to M*e*, by reaſon of the Similitude of the Arches; and as BD to IB, ſo alſo is DA (=L*e*) to H*e*, by *Conſect.* 1. *Def.* 16. Therefore the Tangent H*e*, and the Arch M*e* (to which the ſame L*e* bears the ſame Proportion) will be equal; which is again abſurd. Wherefore BD is to DA, not as DA to a greater A*f*, or to a leſs A*e*; therefore as DA to AE. Q. E. D.

CON-

CONSECTARIES.

I. WHereſore, from what we have deduced, it is evident, if, by means of the Baſe of the Quadratrix AE, you draw a Quadrant, that the ſide of the Quadratrix DA will be equal to it; and conſequently it is the double of the Semi-periphery, and the quadruple of the whole Periphery.

II. It is evident alſo, that you may obtain a right Line equal to the Quadrant DB of any given Circle, if, having deſcribed a Quadratrix, you make, as AE to AD, ſo AD to a third Proportional, equal to the Quadrant DB: Which third Proportional taken four times, will be equal to the whole Periphery.

III. You may alſo obtain a right Line equal to any Arch leſs than a Quadrant, if you make, as DA to H*e*, ſo a third Proportional found (*i. e.* the Quadrant DB) to a fourth, by virtue of *Conſect.* 1. *Def.* 16.

IV. The Quadrature of the Circle therefore, by virtue of *Conſect.* 2. *Def.* 15. *lib.* 1. as likewiſe the Triſection of an Angle, by virtue of *Conſect.* 2. *Def.* 16. *lib.* 2. may be geometrically obtain'd, if the Quadratrix might be number'd among geometrical Curves.

SCHOLIUM II.

C*LAVIUS* was alſo of this Opinion, in the Book aforemention'd, who thought that if the Quadratrix be excluded out of the Number of geometrical Curves; by the ſame reaſon you may alſo exclude the Ellipſis, Parabola, and Hyperbola, ſince they, as well as this, are commonly deſcribed thro' innumerable Points. But, by that great Man's leave, we may deny this Conſequence, by the ſame reaſon as *Des Cartes* has deny'd the Converſe of it in his *Geom. p.* 18, and 19. by virtue of which, he ſuſpects the Antients took the conick Sections, *&c.* for mechanick or non-geometrick Lines, becauſe they did the *Spiral, Quadratrix*, &c. for ſuch. But this is the difference between the Deſcription of the Quadratrix, and the conick Sections thro' Points, that all and every of the Points of the conick Sections, relating to any given Point of the Axis, may be geometrically determin'd; but all the Points of the Quadratrix, promiſcuouſly related to any Point of the generating Quadrant, cannot be geometrically determined; but only thoſe which reſpect ſome certain Point, from which the Quadrant may be divided into two

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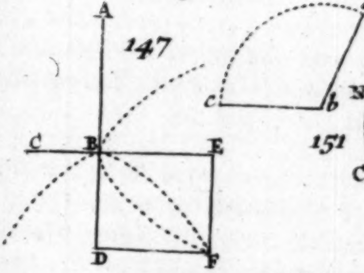

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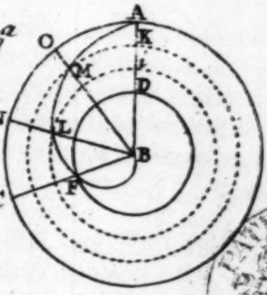

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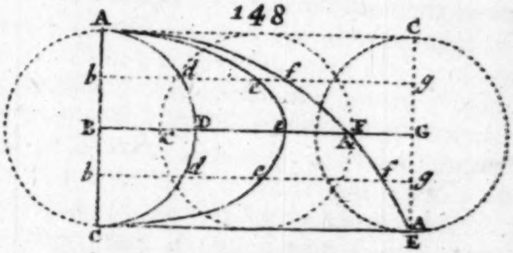

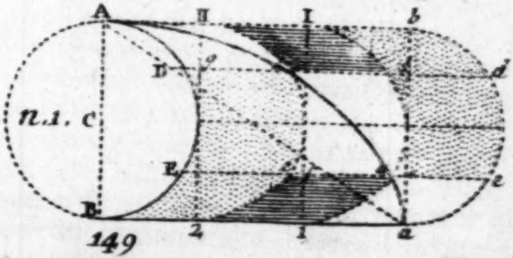

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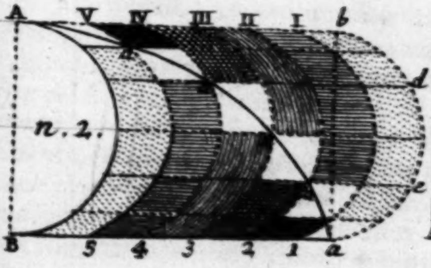

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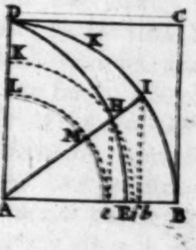

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two Arches of known Proportion. For if, *e.g.* in the Quadrant BD the Point X be given at pleaſure, it will be impoſſible, by *Clavius*'s Rule, to define a Point of the Quadratrix anſwering to it, becauſe the Proportion of the Arches DX and BX is unknown, and conſequently neither can a proportional Section of the right Line AD be made: Not to mention that the laſt Point E (which is the primary and moſt neceſſary one to the Quadrature) even by *Clavius*'s own confeſſion cannot be geometrically defined. We may paſs the like Judgment on *Archimedes*'s Spiral, and ſuch like Curves, which are conceiv'd to be deſcrib'd by two Motions independent on one another; as will be manifeſt to any one who compares the Geneſis of the Spiral with that of the Quadratrix, and what we have hitherto ſaid. Whence neither will *Monantholius*'s Triſection of a given Angle (which he eſſays) by means of a Spiral be enough geometrical; which, in his Book *de Puncto*, *cap.* 7. *p.* 24. he attempts to perform thus: To the Center of a deſcribed Spiral, and its firſt Line BA (*Fig.* 151.) he applies the Angle ABC equal to the given one *abc*; then having drawn Circles thro' F and A, wherein the Legs of the Angle cut the Spiral, he divides the intermediate Space DA into three equal Parts in *i* and K: And then thro' theſe Points he draws Circles cutting the Helix in L and M; and laſtly, having drawn BLN, BMO, he eaſily demonſtrates from the Geneſis of the Spiral, that the Arches AO, ON, NC, are equal. And ſo after the ſame manner, not only any Angle or Arch, but the whole Periphery, may be geometrically divided into as many equal Parts as you pleaſe; only ſuppoſing that this Spiral Line may be number'd among geometrical ones; as we have heretofore hinted, that the Cycloid, Conchoid, Ciſſoid, and logarithmical Curve, *&c.* might be; and we have, above fixteen Years ago, declared our Opinion for it in our *German* Edition of *Archimedes*; and now are therein confirm'd by thoſe celebrated Mathematicians *Leibnitz*, *Craige*, &c. who number Lines of this kind, altho' they cannot be expreſſed by our common Equations among geometrical ones, notwithſtanding the contrary Opinion of *Des Cartes*, &c. becauſe they admit of Equations of an indefinite or tranſcendent Degree, and are capable of a *Calculus*, as well as others, tho' it be of a nature and kind different from that commonly uſed. See the *Acta Erud. Lipſ. ann.* 84. *p.* 234. and *ann.* 86. *p.* 292, and 294.

CHAP. VI.

The Concluſion, or Epilogue of the whole Work.

NOW we may at length underſtand what *Honoratus Faber* delivers concerning the diſtribution of figurate Magnitudes into certain Claſſes, in his *Synopſis Geom. p.* 57.

1. *The firſt Claſs* contains Figures of equal Elements, or Indiviſibles, ſuch as, 1. All Parallelograms; as the Square, Oblong, Rhombus and Rhomboid, the Elements whereof are equal right Lines, as appears from *Def.* 12. *lib.* 1. 2. Convex or concave Surfaces, the Elements whereof are curve Lines moved thro' right Lines by a parallel Motion; among which are chiefly reckon'd cylindrical Surfaces, whereof ſee *Def* 16. *lib.* 1. about the end. 3. Parallelepipedons; and among them the Cube, whoſe Indiviſibles are Squares, or other Parallelograms. 4. Priſms made by the Motion of a Triangle, Trapezium, or any polygonous Body, along a right Line, all the Indiviſibles whereof are conſequently ſimilar, and equal to the generating Plane.

2. *The ſecond Claſs* contains Figures whoſe Elements decreaſe in a ſimple arithmetical Progreſſion; ſuch are, 1. Triangles, as is evident from *Prop.* 37. *lib.* 1. 2. The Circle, and its Sectors, as being reſolvible into concentrick Peripheries, according to *Conſect.* 1, and 3. of the aforecited *Prop.* 3. The Cylinder, as being reſolvible into concentrick cylindrick Surfaces, as its Indiviſibles. 4. The Surface of a Cone, whoſe Elements are circular Peripheries; and alſo of the Pyramid, whoſe Indiviſibles are ſimilar angular Peripheries, every where increaſing in arithmetical Progreſſion. 5. The *parabolick Conoid*, whoſe Indiviſibles are Circles decreaſing, according to the Proportion of the Abſciſſæ in arithmetical Progreſſion, by virtue of *Prop.* 14. *lib.* 2. &c.

3. *The third Claſs* contains elementary Figures, increaſing in a duplicate arithmetical Progreſſion; ſuch are, 1. The Pyramid and Cone; the firſt whereof may be reſolved into angular Planes, the ſecond into circular ones, increaſing according to a Series of ſquare Numbers; as is evident from *Prop.* 38. *lib.* 1. and its Conſectary. 2. The trilinear parabolick

bolick Space, as defin'd *Prop.* 10. *lib.* 2. by the Letters E*h*, HK. 3. The Sphere, as far as it may be reſolved into ſpherical concentrick Surfaces, every one whereof may be conſider'd as a Baſe, taking the Semidiameter for the Altitude. 4. The Cone, as being reſolvable into parallel conical Surfaces, deſcrib'd by the parallel Indiviſibles of a Triangle. 5. The Remainder of a Cylinder after an Hemiſphere of the ſame Baſe and Altitude is taken out, according to *Schol.* 1. of *Prop.* 39. *lib.* 1.

4. *The fourth Claſs* comprehends all Magnitudes reſolvable in Elements, or Indiviſibles increaſing in a triplicate, quadruplicate, *&c.* arithmetical Progreſſion; ſuch we have not treated of, but may be found among Planes terminated by Curves of ſuperior Genders. See *Faber*'s Synopſis, p. m. 67.

5. *The fifth Claſs* is of thoſe Magnitudes, whoſe Indiviſibles decreaſe, proceeding from a ſquare Number, by the odd Numbers, as, 36, 35, 32, 27, 20, 11, *&c.* ſuch are, firſt, an *Hemiſphere*, as is evident from *Prop.* 39. *lib.* 1. 2. An *Hemiſpheroid*, as in *Prop.* 15. *lib.* 2. 3. A *Semi-parabola*, as may be gather'd from the Demonſtration of *Prop.* 10. *lib.* 2. For ſince the Indiviſibles of the circumſcribed trilinear Figure *eh* are found in the duplicate arithmetical Progreſſion $\frac{1}{4}$, $\frac{4}{4}$, $\frac{9}{4}$, $\frac{16}{4}$, *&c.* the Indiviſibles of the Semi-parabola will neceſſarily be $\frac{16}{4}$, $\frac{15}{4}$, $\frac{12}{4}$, $\frac{7}{4}$, *&c.*

6. We may make *a ſixth Claſs* of thoſe Magnitudes, whoſe Indiviſibles decreaſe in a like Progreſſion, not of the Numbers themſelves, deſcending by the odd Numbers from a given Square, but of their Roots, which are for the moſt part ſurd ones; ſuch as is the *Semi-circle*, as is evident from *Prop.* 43. *lib.* 1. and *Prop.* 5. *lib.* 2. and alſo the *Semi-ellipſis*, &c.

7. *The ſeventh Claſs* comprehends thoſe Magnitudes, whoſe Elements are in a Progreſſion of a double Series of Numbers, as in the *Hyperbolick Conoid*, as may be ſeen in the *Scholium* of *Prop.* 6. *lib.* 2.

But we ſhall omit the other Claſſes of Magnitudes of a ſuperiour Gender, the Conſideration whereof theſe Elements either have not touch'd on, or only by the by, (which any one who pleaſes may ſee in *Faber*'s Synopſis, eſpecially thoſe which he comprehends under the ſixth and ſeventh Claſſes, *p.* 70. and the following) and ſhall only note two Things here. 1. That ſince in the firſt Claſs we place Parallelograms and Cylinders ; in the ſecond, Triangles ; in the third, Pyramids and Cones ; in the fifth, Hemiſpheres ; in the ſixth, Semicircles, *&c.* We may, with *Hon. Faber*, call the firſt Claſs, that of cylindrical or *Parallelogrammatick Figures* ; the ſecond, the Claſs of *Triangular Figures* ; the third, of *Pyramidals* ; the fifth, of *Hemiſpherical Figures* ; the ſixth, of *ſemicircular ones*, &c. 2. That having after this manner ranged or reduced homogeneous Figures, or thoſe of like condition, to a few Claſſes ; their Dimenſion, and conſequently almoſt the whole Buſineſs of meaſuring, may be very compendiouſly reduc'd to a few Rules ; whereof we will here give the Reader a ſhort Specimen, in the following

CONSECTARIES.

I. THE Dimenſion of Parallelogrammatick Figures, *i. e.* of thoſe of the firſt Claſs, may be had, by multiplying the whole Baſe by the whole Altitude : See *lib.* 1. *Def.* 12. *Conſect.* 7. *Def.* 18. *Conſect.* 6. *Def.* 16. *Conſect.* 3, and 4.

II. The Dimenſion of triangular Figures, *i. e.* of thoſe of the ſecond Claſs, may be had by the Multiplication of the whole Baſe, by half the Altitude, or of half the Baſe by the whole Altitude ; [ſee, *lib.* 1. *Def.* 12. *Conſect.* 8. *Def.* 15. *Conſect.* 2. *Def.* 18. *Conſect.* 4. *lib.* 2. *Prop.* 14.] and their Proportion is to their reſpective Parallelograms of the ſame Baſe and Altitude as 1 to 2 ; [beſides the *Prop.* already cited, ſee, *lib.* 1. *Prop.* 37. and its firſt *Conſectary.*]

III. The Dimenſion of Pyramidals, *i. e.* of Magnitudes of the third Claſs, may be obtained by the Multiplication of the Baſe, by the third Part of the Altitude ; [ſee, *lib.* 1. *Definit.* 17. *Conſect.* 3, and 4. and *Def.* 20. *Conſect.* 1, &c.] and their Proportion to the correſponding Figures of the firſt Claſs of the ſame Baſe and Altitude, is as 1 to 3. [Beſides the *Prop.* already cited, ſee *Prop.* 38. *lib.* 1. and its *Conſect.* *Prop.* 39. and *Scholium* 1. *lib.* 2. *Prop.* 10, &c.]

IV. The Proportion of Hemiſpherical Figures, *i. e.* of the fifth Claſs to the correſponding ones of the firſt Claſs of the ſame Baſe and Altitude, is as 2 to 3 ; [See, *lib.* 1. *Prop.* 39. *lib.* 2. *Prop.* 10, and 15.] and ſo their Dimenſion may be had by multiplying their Baſe by $\frac{2}{3}$ of their Altitude.

V. The Proportion of ſemicircular Figures, *i. e.* of thoſe of the ſixth Claſs to ſo many correſponding ones of the firſt Claſs of the ſame Baſe and Altitude, cannot be expreſſed by whole Numbers, or by a ſmall Fraction; [See, *lib.* 1. *Prop.* 43. and *lib.* 2. *Prop.* 11.] and conſequently their exact numeral Dimenſion cannot be had.

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INTRODUCTION

TO

SPECIOUS ANALYSIS;

OR,

The New Geometry, chiefly according
to the Method of DES CARTES;

But much Facilitated by later Inventions, *&c.*

By *J. CHRIS. STURMIUS.*

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Preface to the READER.

S*INCE thoſe ingenious Mathematicians of this preſent Age, which is now drawing to a Concluſion,* viz. Vieta, Ougthred, Harriot, des Cartes, Schooten, Beaune, Van Hudde, Heuraet, de Witte, *and* Sluſius, *and ſeveral other Famous Men coeval with them, have, by their Endeavours, improved the* Algebra *of the Antients, raiſed it to vaſtly an higher pitch, and brought it from Numbers to univerſal Symbols, and not only found the excellent Uſes of it in Geometry themſelves, but alſo communicated them to others: Almoſt all Countries have furniſh'd us with ſome excellent Perſons, who, treading in the footſteps of their Predeceſſors, have endeavour'd to advance it yet further; and even our Times are not without thoſe of the higheſt rank, as* Wallis, Baker, Renaldinus, Mengolius, Huygens, Malbranch, Leibnitz, Craan, *and ſeveral others, who endeavour to promote this Science, deſervedly reputed the very* Apex *of Human Reaſon, and carry it more and more towards its utmoſt Perfection, by daily augmenting it with new and curious Inventions. But in the mean time, while theſe ingenious Men wholly buſy themſelves in promoting it, there are few found who condeſcend to explain the firſt Principles of it, and ſhew a ready way to young Beginners to arrive at the knowledge of thoſe Inventions. It is not long ſince a certain Friend of mine, who for ſome time has publickly and ſucceſsfully taught theſe Sciences, complain'd to me by his Letters, of the want of a good Guide to Specious*

Analyſis, whereby he might inſtil the Principles of that admirable Art into his Auditors. Whoſe Deſires, being not juſt then at leiſure to ſatisfy, I immediately after projected this Introduction, which at length you ſee finiſhed in the Form we now preſent you with it, wherein we have all along conſulted to ſuit the Endeavours of young Beginners, as far as poſſible, as we thought our ſelves engag'd by the Duties of our Profeſſorſhip to do; and have compriz'd the Precepts of the Art in ſix or ſeven Pages, and accommodated Examples of every kind to illuſtrate them; wherein if I have but indifferently accompliſh'd my Deſign, I ſhall not think my Labour loſt. We here add it to our Matheſis Enucleata, *both as being properly a part of it, and more eſpecially, becauſe this Introduction preſuppoſes the Reader to be acquainted with the firſt Principles of ſpecious Computation, which we have therein laid down. And ſo we commit our Endeavours to the Peruſal and Cenſure of the candid Reader.*

INTRODUCTION

TO

SPECIOUS ANALYSIS.

THE *Analytick Art*, or *Specious Analyſis* is ſolely ſubſervient to finding of Theorems, and reſolving Problems, by leading us from certain *Data* or given Quantities, into the knowledge of unknown and ſought ones, by a Chain of certain and infallible Conſequences: This admirable Artifice may be reduced to four Primary Heads, *viz. Denomination*, *Reduction*, *Equation*, and *Effection* or *Conſtruction*, *if the Problem be a Geometrical one.*

I. *DENOMINATION.*

BY Denomination is underſtood a preparatory Impoſition of Names peculiar to each Quantity, whereby every one of the Quantities given or ſought, are denoted by one or more peculiar Letters of the Alphabet at pleaſure; but with this (arbitrarious) difference, that known or given Quantities are mark'd by the former Letters of the Alphabet, *a*, *b*, *c*, *&c.* and the unknown or ſought ones by the latter, *z*, *y*, *x*, *&c.* But although this Impoſition of Names, is, as we have ſaid, altogether arbitrarious, yet there often happens not a little Facility to the Solution itſelf, by its being choſen as accommodate as poſſible to the Conditions of the Quantities given and ſought ; which any one will learn better by Uſe than Precepts. As we find *e.g.* that both Theorems may be demonſtrated, and Problems reſolv'd, by an extraordinary Compendium, if we denote any Ratio of two given homogeneous Quantities by a, and $e a$, b and $i b$, d and $o d$, &c. (*viz* by expreſſing the Names of the Ratio's by e, and i, and o, &c.) and continued proportionality, by a, ea, e^2a, e^3a, &c. and diſcontinued or diſcrete, by b, $i b$; c, $i c$; d, $i d$; or after the like manner, as we have done in our *Math. Enucl. Lib.* 1. *Cap.* 2, 3, 4, 7. and *Lib.* 2, *Cap.* 1. &c.

II. *EQUATION.*

HAVING thus given each Quantity its Name, and making no farther diſtinction between the Quantities given and thoſe ſought, but treating them all promiſcuouſly, and as already known ; you muſt carefully ſearch into and diſcuſs all the Circumſtances of the Queſtion, and making various Compariſons of the Quantities, by adding, ſubſtracting, multiplying, and dividing them, &c. 'till at length, which is the chief Aim and Deſign of it, you can expreſs one and the ſame Quantity two ways, which is that we call an *Equation* : And you muſt find as many of theſe Equations, or Equalities of literal Quantities, (as expreſſing the ſame thing) as there are ſeveral unknown Quantities in the Queſtion, independent on each other, and conſequently denominated by ſo many different Letters, z, y, x, &c. But if ſo many Equations cannot be found, after having exhauſted all the Circumſtances of the Queſtion by one or two Equations ; that is a ſign the other unknown Quantities may be aſſumed at pleaſure : Which the Examples we ſhall hereafter bring, will more fully ſhew.

But as here alſo (as likewiſe in all this Art) Ingenuity and Uſe do more than Rules and Precepts ; yet we will here ſhew the principal Fountains, for the ſake of young Beginners ; from whence Equations, according to Circumſtances obvious in the Queſtion, are uſually had. Theſe are partly Axioms ſelf-evident, *e.g.*

That the Whole is equal to all its Parts taken together.

That thoſe Quantities which are equal to ſome third, are equal among themſelves.

That the Products or Rectangles under the Parts or Segments, are equal to the Product of the Whole.

Partly ſome univerſal Theorems that are certain and already demonſtrated ; as, *That three * continual Proportionals being propoſed, the Rectangle of the Extremes is equal to the Square of the Mean.* * Eucl. 6. 17.

† *Four being propoſed, whether in continued or diſcontinued Proportion, the Product or Rectangle of the Extremes is equal to that of the Means.* † lib. 6. 16.

And ſeveral others ſuch like, which we have demonſtrated in *Cap.* 2d, 3d, and 4th. *Lib.* 1. of our *Matheſis Enucleata.*

In the laſt place, partly ſome particular Geometrical Theorems already demonſtrated, as *e. g.* that common Pythagorick one,

*That in right-angled Triangles * the Square of the Hypothenuſe is equal to the two Squares of the Sides.* * lib. 1. 47.

That the Square of the Tangent of a † Circle, is equal to the Rectangle of the Secant, and that Segment that falls without the Circle; the firſt whereof we have demonſtrated, *Lib. Math. Enuc. Def.* 13. *Schol.* and alſo *Prop. Conſect.* 8. alſo *Prop,* 44. after various ways; to which may be numbred *Prop.* 34. with *Schol.* 2. *n.* 3. *Prop.* 37. *&c. Prop.* 45. and 46. alſo the 48th, and ſeveral others in *Lib.* 1. *Math. Enucl.* and likewiſe *Lib.* 2. *Prop.* 1, 2, 3, and ſeveral following ones. And as for Examples both of Denomination, and Equations found after various ways, you may ſee them hereafter follow, and ſome we will here give you by way of Anticipation. † lib. 3. 36.

III. *REDUCTION.*

AN Equation thus found muſt be reduced, *i. e.* thoſe two equal Quantities, which for the moſt part are very much compounded of the Quantities given and ſought together, muſt be reduced to ſuch a form, by adding or ſubſtracting ſomething to or from each part, or multiplying and dividing by the ſame, *&c.* that the unknown or ſought Quantity alone, or its Square, or Cube, or Biquadrate, *&c.* may be found on the one ſide, and on the other, the Quantity expreſs'd by meer given or known Letters, or affected with the unknown and ſought ones; as are theſe Forms which follow, diſtinguiſh'd by their Names prefix'd to them.

A ſimple Equation, $z = b$, or $y = \frac{ab}{c}$

A pure Quadratick, $yy = ab$ or $xx = \frac{aa - bb}{2}$

A pure Cubick, $z^3 = abc$, or $y^3 = \frac{a^3bb}{cd}$

An affected Quadratick, or, $\begin{array}{l} x^2 = -ax + b^2 \\ y^2 = 2by \; \frac{+\,2abc}{d} \end{array}$

An affected Cubick, $z^3 = az^2 + b^2z - c^3$ &c.
A Biquadratick, $y^4 = ay^2 + b^2y^3 - c^3y + d^4$ &c.

When you have reduced your Equation firſt found to one of theſe Forms, or ſome other like them, there are Rules in readineſs, whereby the value of the unknown or ſought quantity z, or y, or x, may be either expreſs'd in Numbers, if the Queſton be an Arithmetical one, or geometrically determined, if it be a Geometrical one, which is that we call *Effection* or *Conſtruction*.

Thus therefore the whole, or at leaſt the chief Buſineſs of Analyticks is converſant in finding a convenient or fit Equation: For Reduction is very eaſy, as conſiſting only in eaſy Operations and mere Axioms; as *e. g.*

If to or from equal Quantities you add or ſubſtract equal ones, the Aggregates or Remainders will be equal.

If equal Quantities are multiplied or divided by the ſame quantity, the Products or Quotients thence ariſing will be equal, &c.

IV. *EFFECTION*, or *CONSTRUCTION*.

1. *In ſimple Equations.*

SUppoſe $z = b$, the quantity b itſelf is that ſought.

$$\left.\begin{array}{l} 2.\ \text{If } z = \frac{ab}{c} \cdots \\ \text{or } x = \frac{aa}{b} \cdots\cdots \\ \text{or } y = \frac{fb+fg}{b-1} \\ \text{or } z = \frac{fb-fg}{b+1} \end{array}\right\}\text{make}\left\{\begin{array}{l} \text{as } c \text{ to } b, \text{ ſo is } a \text{ to } z, \\ \text{as } b \text{ to } a, \text{ ſo is } a \text{ to } x, \\ \text{as } b-1 \text{ to } b+g, \text{ ſo is } f \text{ to } y, \\ \text{as } b+1 \text{ to } b-g, \text{ ſo is } f \text{ to } z, \text{ \&c.} \end{array}\right.$$

every where according to *n.* 2. Schol. 2. Prop. 34. Lib. 1. *Math. Enuc.*

3. If z be $= \frac{kl+mn}{r+s}$ the Reſolution of it into Proportionals will be more difficult, becauſe neither of the Letters are found twice in the Numerator. That therefore you may have for Ex-

Example, k twice, you muſt make as k to n, ſo is m to a fourth Proportional, which call p; then by virtue of *Prop.* 18. *Lib.* 1. will $kp=mn$, and the propoſed Equation will be changed into this $z=\frac{kl+kp}{r+s}$, to be now conſtructed from the 2*d* Caſe.

Or if $y=\frac{kl+mn}{r-s}$, find a mean Proportional between k and l, which call p; and between m and n, which call q, according to *n.* 3 of the afore-cited Schol. and the propoſed Equation, by virtue of *Prop.* 17. will be in this Form: $y=\frac{pp+qq}{r-s}$, therefore in the right-angled △ (*Fig.* 1.) make $AB=p$ and $BC=q$; and the □ of AC by virtue of the Pythagorick Theorem will be $=pp+qq$; which ſince it muſt be divided by $r-s$, make further, by *Prop.* 18. as $r-s$ to the $\sqrt{pp+qq}$, ſo is $\sqrt{pp+qq}$ to y, according to the aforecited.

4 In like manner if x be $=\frac{bg-mn}{c+d}$, firſt make as b to m, ſo is n to a fourth, which call k; and ſo putting bk for mn, the Equation will be reduced to the ſecond Caſe under this form: $x=\frac{bg-bk}{c+d}$.

Or thus: Find a mean Proportional between b and g, which call p, and between m and n, which call q; and the propoſed Equation will be in this Form: $x=\frac{pp-qq}{c+d}$ therefore (in *Fig.* 2.) make $AB=p$, and having deſcribed a Semicircle on it, apply $BC=q$; then by virtue of *Schol* 5. *Prop.* 34. □AC will be $=pp-qq$; which ſince it muſt be divided by $c+d$, make farther as $c+d$ to $\sqrt{pp-qq}$, ſo is $\sqrt{pp-qq}$ to x; all from the ſame Foundations, whence you have the Conſtruction of the third Caſe.

5. If z be $=\frac{aa\,bc}{ffg}$ firſt make as f to a, ſo is a to a third Proportional m; and putting fm for aa, you will have $z=\frac{fmbc}{ffg}$, i. e. $\frac{mbc}{fg}$: Secondly make as f to m, ſo is b to a fourth n, and by putting fn for mb, you'll have $z=\frac{fnc}{fg}$, i. e. $\frac{nc}{g}$, wherefore thirdly as g to n, ſo is c to z.

6. If y be $= \frac{hll}{mm}$ firſt make m to n, as l to a fourth, which call n, and by putting mn for hl, you'll have $\frac{mnl}{mm}$ i. e. $\frac{nl}{m} = y$. Therefore ſecondly, as m to n, ſo is l to y by Caſe 2: So that the Conſtruction of the fifth and ſixth Caſes is nothing but reiterations of the Rule of Three, according to what we have often inculcated, *N. 2d.* & *3d. Schol.* 2. *Prop.* 34*th*.

2. *In ſimple Quadratick Equations.*

1.

IF $\left.\begin{array}{l} xx = ab \\ \text{or } y^2 = 1c \\ \text{or } z^2 = \frac{3}{4} dd \end{array}\right\}$ you'll have $\left\{\begin{array}{l} x = \sqrt{ab} \\ y = \sqrt{1c} \\ z = \sqrt{\frac{3}{4} dd} \end{array}\right.$ that is, to a mean proportional between $\left\{\begin{array}{l} a \text{ and } b \\ 1 \text{ and } c \\ \frac{3}{4} d \text{ and } d \end{array}\right.$ and ſo the Conſtruction will be had from *n.* 3. *Schol.* 2. *Prop.* 34. (See *Fig.* 3.)

2. If $\left.\begin{array}{l} y^2 = fg + kl \\ \text{or } x^2 = fg - kl \end{array}\right\}$ you'll have $\left\{\begin{array}{l} y = \sqrt{fg + kl} \\ x = \sqrt{fg - kl} \end{array}\right.$ make therefore for the one, the Right-angled Triangle ABC (*Fig.* 4.) whoſe Side

AB is = to a mean Proportional between f and g,

And BC is = to a mean Proportional between k and l;

and for the other, a Right-angled △ (*Fig.* 5.) whoſe ſide AB is = to a mean Proportional between f and g, and the ſide BC = to a mean Proportional between k and l;

$\left.\begin{array}{l} \text{and on the one hand,} \\ \text{the Hypothenuſe} \\ \text{on the other, the Side} \end{array}\right\}$ AC will be the value of $\left\{\begin{array}{l} y \\ \\ x \end{array}\right.$ ſought

And all by virtue of the Pythag. Theor. and according to *Schol* 5. *Prop.* 34. or the Conſectaries of *Prop.* 44. See *Fig.* 4. and 5.

3. If $z^2 = \frac{fhhkk}{fcc}$ i. e. $\frac{hhkk}{cc}$ extracting the Roots on both ſides, and z will $= \frac{hk}{c}$ and ſo this will be the ſecond caſe of ſimple Equations.

4. If y^2 be $= \frac{fghk}{lm}$, make firſt as l to f, ſo g, to a fourth,

which call n, and by putting ln for fg, you'll have $y^2 = \frac{lnhk}{lm}$ i. e. $\frac{nkh}{m}$. Make Secondly, as m to n, ſo h to a fourth, which call p; and by putting mp for nh, you'll have $y^2 = \frac{mpk}{m}$ i. e. pk, and ſo this is brought to the firſt caſe of the preſent Equations.

5. If $x^2 = \frac{qbcc + bccd + qbcd + bcdd}{fg + lm}$; in the firſt place, the Rectangles fg and lm being turned into Squares, and collected into one Sum, make them $= nn$. Then (ſince cc and cd are multiplied by $qb + bd$) in like manner let qb and bd added make pp; and you'll have $x^2 = \frac{ppcc + ppcd}{nn}$. Thirdly, (ſince pp is already multiplied by $cc + cd$) having added $cc + cd$ into one Sum, which let be rr; then x^2 will $= \frac{pprr}{nn}$, and ſo it will be the third caſe of the preſent Equations.

3. *In affected Quadratick Equations.*

IF zz be $= az + bb$, then will $z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$; which may be thus in ſhort demonſtrated *a priori*: Since $z^2 - az = bb$ *per Hypoth.* and $\frac{1}{4}aa$ be added to that firſt quantity $z^2 - az$, it becomes an exact Square, the Root whereof is $z - \frac{1}{2}a$; therefore $z^2 - az + \frac{1}{4}aa = bb + \frac{1}{4}aa$, and conſequently $z - \frac{1}{2}a = \sqrt{bb + \frac{1}{4}aa}$; and laſtly $z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$, or $\frac{1}{2}a - \sqrt{\frac{1}{4}aa + bb}$; which laſt Root is a negative one, and leſs than nothing, but yet gives you the propoſed Equation back again as well as the former; as will be evident to any one who tries, *viz.* having transferr'd $\frac{1}{2}a$ on the other ſide, and ſo the two equal Quantities $z - \frac{1}{2}a$ and $-\sqrt{\frac{1}{4}aa - bb}$ being ſquared. For here will come out $\frac{1}{4}aa + bb$ as well as if the radical Sign were affected with the Sign + becauſe − by − gives +. Therefore $zz - az + \frac{1}{4}aa = \frac{1}{4}aa + bb$, and taking away on both ſides $\frac{1}{4}aa$, $zz - az + bb$, i. e. $zz = az + bb$.

Therefore the value of this Root will be had geometrically, by making (in *Fig. 6.*) CD $= \frac{1}{2}a$, and DE perpend. $= b$; that ſo the Hypothenuſe CE may be $\sqrt{\frac{1}{4}aa + bb}$; and moreover drawing out on both ſides CD, and with the inverval CE deſcribing a Semi-circle AEB: This being done, AD will be the Value ſought of the affirmative Root z, and DB of the negative one. Or with *Des Cartes* (*Fig.* 7.) making CD $= \frac{1}{2} a$, and DE $= b$; then ſhall CE $= \sqrt{\frac{1}{4} aa + bb}$, and conſequently AE ſhall be the Value of the affirmative Root z, and BE of the negative one.

2. If y^2 be $= -ay + bb$, then will $y = -\frac{1}{2} a + \sqrt{\frac{1}{4} aa + bb}$; which again may thus apear: Since $y^2 + ay$ is $= bb$, by *Hypoth.* adding to both ſides $\frac{1}{4} aa$, and the firſt Quantity will be an exact Square, and $y^2 + ay + \frac{1}{4} aa = \frac{1}{4} aa + bb$. Therefore the Roots will be alſo equal, *viz.* $y + \frac{1}{2} a = \sqrt{\frac{1}{4} aa + bb}$; and conſequently $y = \sqrt{\frac{1}{4} aa + bb} - \frac{1}{2} a$, or $-\sqrt{\frac{1}{4} aa + bb} - \frac{1}{2} a$; which is a negative Root.

The Value of theſe Roots may beh ad geometrically, *viz.* that of the affirmative Root DB in *Fig.* 6. or BE in *Fig.* 7. and of the negative one AD in the 6th *Fig.* and AE in the 7th.

3. If $xx = ax - bb$ you'll have

$$x = \frac{1}{2} a + \sqrt{\frac{1}{4} aa - bb}$$

$$\text{or } \frac{1}{2} a - \sqrt{\frac{1}{4} aa - bb}:$$

Which may be demonſtrated after the ſame way *a priori*, as the former Caſes, *viz.* Since $x^2 - ax$ is $= -bb$, adding on both ſides $\frac{1}{4} aa$; the former Quantity will be an exact Square *viz.* $x^2 - ax + \frac{1}{4} aa = \frac{1}{4} aa - bb$. Therefore the Root of the one ſide $x - \frac{1}{2} a =$ to the Root of the other, *viz.* $\sqrt{\frac{1}{4} aa - bb}$, and adding on both ſides $\frac{1}{2} a$, x will be $= \frac{1}{2} a + \sqrt{\frac{1}{4} aa - bb}$ which is one of the affirmative Roots. Or $\frac{1}{2} a - \sqrt{\frac{1}{2} aa - bb}$, which in this caſe is alſo an affirmative one. But the Value of each may be obtain'd by making (*Fig.* 8.) CB $= \frac{1}{2} a$, and by erecting BD perpendicularly $= b$, and making the Semi-circle BEA, and drawing DE parallel to CB, and letting fall the Perpendicular EF: For thus CF will be $\sqrt{\frac{1}{4} aa - bb}$; and conſequently AF $\frac{1}{2} a + \sqrt{\frac{1}{4} aa - bb}$, and FB $= \frac{1}{2} a - \sqrt{\frac{1}{4} aa - bb}$.

Or, with *Des Cartes*, making (*Fig.* 9) CB $= \frac{1}{2} a$, and BD $= b$, drawing DF parallel to CB, that ſo FD may be one Root, and ED the other; as is manifeſt from the precedent Conſtruction, and its Rule. See alſo another Deduction from *Des Cartes*'s Conſtructions, *Schol.* 1. *Prop.* 47. *Lib.* 1. *Math. Enucl.*

N B. 1. *The ingenious* Schooten *has before ſhew'd this Method of demonſtrating, and alſo of finding out theſe Rules in* *his Comment on the Geometry of* Des Cartes, p. m. 163. *and moreover deduces another ingenious Method for all the three Caſes of theſe Equations, by taking away the ſecond Term in the Equation,* p. 290. *and the following, where we may make only this Remark concerning the third Caſe; that perhaps the Rule might be better deduced, if we make* $x = \frac{1}{2}a - z$ *rather than* $x = z - \frac{1}{2}a$.

N. B. 2. *If any one has a mind to ſee the new Conſtructions of affected quadratick Equations of the Abbot* Catelan, *he may find them in* Acta Erud. Lipſ. Ann. 1682. p. 86. *and in the 27th Journal* des Scavans, 1 Dec. 1681.

IV. *For Cubick and Biquadratick Equations both ſimple and affected, and alſo for all before mentioned, and conſequently univerſally for all not exceeding the fourth Dimenſion.*

THE Value of the unknown Quantity or Root, may for any Caſe be determined by one general Rule, found out by Mr. *Thomas Baker* an *Engliſh-man*, occaſioned by what *Des Cartes* had taught concerning this Matter, *lib.* 3. *Geom. p.* 85. and the following; but now very much perfected by this Rule, and made more ſimple. Now that this Rule may be the better comprehended by Learners, we will premiſe theſe following things.

1. That all Equations occurring under thoſe Forms, or the like, which we have before ſhewn in the Article of Reduction, muſt always for this purpoſe be ſo changed as to have all the Terms or Parts of the Equation both known and unknown, affected and not affected, brought over to one ſide promiſcuouſly, and ſo on the other there will ſtand o or nought, as, *e. g.*

let $x - b$ be $= o$, or $y - \frac{ab}{c} = o$, or

$yy - ab = o$, or $xx - \frac{aa + bb}{2} = o$, or $z^3 abc = o$, or

$y^3 - \frac{a^2 bb}{ca} = o$, or $x^2 + ax - b^2 = o$, or

$y^2 - 2by - \frac{2abc}{d} = o$, or

$$z^3 - az^2 - b^2z + c^3 = 0, \text{ or}$$
$$y^4 - ay^3 - b^2y^2 + c^3y - d^4 = 0, \text{ \&c.}$$

which alſo was uſual to *Des Cartes*, in *Lib.* 3.

2. In all Equations, the known Quantity or Co-efficient of the ſecond Term we will generally denote by the Letter p; that of the third Term, by the Letter q; of the fourth, by r; and of the fifth (or abſolute Number) by S; according to *Des Cartes*, but with ſome little Alteration: So that hence the Equations we have before been treating of, and all others like them (every where denoting the unknown Quantity by x) may all be reduced to theſe Forms:

$$x - p = 0$$
$$x^2 + px - q = 0$$
$$x^3 - px^2 - qx + r = 0$$
$$x^4 - px^3 + qx^2 - rx + S = 0, \text{ \&c. \&c.}$$

3. Theſe and the like Equations may either occur whole, or with all their Terms, as here; or deprived of one or more of them, as the following Examples will ſhew, where we will always put an Aſteriſk in the place of the deficient Term.

$$x^2 * - q = 0$$
$$x^3 ** + r = 0$$
$$x^4 *** - S = 0$$
$$x^3 \mp px * \mp r = 0$$
$$x^4 * \mp qx^2 \mp rx \mp S = 0$$
$$x^4 \mp px^3 \mp qx^2 * \mp S = 0$$
$$x^4 ** \mp rx \mp S = 0$$
$$x^4 \mp px^3 ** \mp S = 0$$

4. The unknown Quantity in any Equation has * as many diverſe Roots or Values, as the Equation has Dimenſions; which *Des Cartes* ſhews, *lib.* 3. *Geom. p.* 69. at the ſame time evidently demonſtrating this, *viz.* that ſome of thoſe Roots may be Negative: Which from him we here ſuppoſe.

When therefore *Des Cartes* in his Conſtruction of cubick and biquadratick Equations, *p.* 85. and the following, requires, as a neceſſary Condition, the Ejection of the ſecond Term in the given Equation, unleſs it were already wanting, and ſo was obliged to ſhew a way to eject it, with ſeveral other Preparations; and afterwards, when by help of his Rule, deliver'd, *p.* 91. he had found a way of finding two mean Proportionals, and dividing any given Angle into three equal Parts, then he uſes it for ſolving other ſolid Problems, or finding two mean Proportionals, or triſecting an Angle. But the general Rule of *Baker* has no need of theſe Methods or Helps, neither of the Ejection of the ſecond Term, nor any other Preparation, but immediately ſhews us a way to find all the Roots of any given Equation, by help of a Circle and Parabola, both affirmative and negative, whether the Equation wants any Term or not, and howſoever affected, after the way we will now, and perhaps a little more diſtinctly, ſhew.

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* *N. This is an Error, as has been proved by* Dr. Wallis.

1. This Rule ſuppoſes with *Des Cartes*, a Parabola NaM to be already deſcribed, (ſee *Fig.* 10, and 11.) whoſe *Latus Rectum* is L, or 1, and its Axis ay; which *Des Cartes* only making uſe of, and never thinking of the other Diameters, was forced to take away the ſecond Term of the Equation, &c. *Baker* therefore (ſtrangely perfecting the *Carteſian* Geometry by this one thought) draws the Ordinate BA $=\frac{p}{2}$ to the Axis ay, if the Quantity p or ſecond Term be in the Equation, *i. e.* at top of the Axis a, on the right hand, he raiſes the Perpendicular $a\mathrm{E}=\frac{p}{4}$, and from E draws EAY parallel to the Axis ay; whereby he obtains the Diameter AY ſought.

2. Having made this Preparation, the whole Buſineſs depends on this, *viz.* to find the Center of a Circle to be deſcribed through the Parabola, which (by virtue of ſome arbitrary Suppoſitions in the beginning) he always ſeeks on the left ſide of the Axis or Diameter, by help of two Lines $\frac{\mathrm{AD}}{a\mathrm{D}}$ or b, and DH or d; *viz.* by placing the former in the Axis from a to D, if p be wanting in the Equation, or in the Diameter AY from A to D, if p be there; and from the Point D drawing a Line perpendicular to aD or AD towards the left hand.

3. He ſhews how to find the Quantity of either of theſe Lines (which is here very requiſite) in any given Equation, by a certain general Rule (which he calls the Central Rule, becauſe it alone helps to find the Center H) comprehended in theſe Terms :

$$\text{Part}\begin{cases}1.\ \frac{L}{2}+\frac{p^2}{8L}\underset{\backsim}{+}\frac{q}{2L}=b=\text{AD or }a\text{D}\\ 2.\ -\frac{p}{4}+\frac{p^3}{16L^2}\underset{\backsim}{+}\frac{pq}{4L^2}\underset{\backsim}{+}\frac{r}{2L^2}=d=\text{DH}\end{cases}$$

4. This Rule, as it ſtands here whole, only anſwers to thoſe Equations wherein are all the Terms p, q, and r; and in the mean time may alſo be eaſily accommodated to all other Caſes, only obſerving theſe things: 1. Whatever of the Quantities p, q, r, be wanting in the propoſed Equation, that muſt alſo be reſpectively omitted, or put out of the general central Rule, that ſo the remaining Quantities may determine the ſpecial or particular central Rule. 2. As for what belongs to the Signs, *viz.* whether $+$ or $\backsim$ (which latter Sign denotes a dubious Caſe, either that the firſt muſt be ſubſtracted from the latter, or contrary-wiſe, as the Matter will bear) muſt be put in the central Rule, he notes that in the Rule you'll always have $+\frac{r}{2L^2}$, unleſs when in the propoſed Equation, p and r are affected with diverſe Signs: By what Sign ſoever in the propoſed Equation it happens that the Quantity q is marked with, it muſt be noted with the contrary one (altho' involv'd with other Quantities) in the Rule; as may be ſeen in the Application of the Rule to all ſpecial Caſes done by the Author himſelf for the ſake of Beginners, and is exhibited in the Synopſis hereunto adjoining, which yet we have thought fit to give at the end of this Treatiſe, much more contract as to the central Rules, in a ſhort Compendium by way of Appendix.

5. By theſe Rules therefore, the Quantities of the Lines aD or AD and DH will be ſo determined, that the Parts in the Rule affected with the Sign $+$ (taken either aggregately or ſingly) will fall downwards from a or A towards y, and on the left hand of D; but the negative Parts, or thoſe affected with the Sign $-$, will fall from a or A upwards, and to the right of the Point D: Which being done, the Center H will be found.

6. From the Center H thro' the Vertex of the Axis a, (if the Point D be found in the Axis) or in the other Caſe, thro' the Vertex of the Diameter A, you muſt draw a Circle; which by cutting or touching the Parabola, will determine the Roots ſought, if the Equation be not a Biquadratick, *i. e.* if it has not the Quantity S: otherwiſe another

Point L or Z muſt be found, (*vid.* Fig. 12, and 13.) and a Circle deſcribed on the Radius HL or HZ, according to *Des Cartes*, *p.* 86. and following.

7. *Viz.* If you have — S, in the Line H*a* or HA produced on each ſide A or *a*, you muſt take $AI = L = 1$, and $AK = \frac{S}{L}$, and deſcribing a Semicircle on IK, draw AL perpendicular to AH, to obtain the Point L. (See *Fig.* 12.) But if you have + S; then in another Semicircle deſcribed on AH, apply the Line $AZ =$ to AL thus found, thereby to obtain the Point Z. (See *Fig.* 13.)

8. A Circle therefore deſcribed from H through *a* or A, if S be wanting, but thro' L if there be — S, and thro' Z if + S, may touch or cut the Parabola either in 1, 2, 3, or 4 Points; from which, if you let fall Perpendiculars to the Axis or Diameter, you will obtain all the Roots of your Equation, both affirmative and negative.

9. And, 1. If *p* be wanting in the Equation, and — *r* be there, the affirmative Roots will be on the left ſide of the Axis, as NO; and the negative ones, as MO, on the right ſide. 2. But if there be *p* in the Equation, and it be negative, the affirmative Roots will fall on the left ſide of the Diameter, and the negative ones on the right; but if *p* be affirmative, on the contrary, the affirmative will be on the right hand, and the negative on the left.

10. But if the Circle neither touches nor cuts the Parabola in any Point, it is a ſign that the Equation is impoſſible, and has no Root either affirmative or negative, but only imaginary ones. All which, how they may be found out, and that they are undoubtedly true, are demonſtrated *a poſteriori*, in an eaſy and plain way by the Author, wherefore we ſhall not give the Demonſtrations of them here; but remit the Reader, after he has made a little Progreſs in this Art, to the Author himſelf.

11. Wherefore now (omitting alſo in this place the Doctrine of the Compoſition of plain and ſolid geometrical *Loci*, or Places, which would ſerve for a Complement of the Analytick Art) we will ſhew the Practice of theſe Rules already delivered, premoniſhing only this from Mr. *Baker*, viz. if the *Latus Rectum* be made Unity, then L in the central Rules and all its Powers may be omitted; and ſo the Rules exhibited more compendiouſly: as we have already done in

our Synopſis, and may be ſeen from the form of a general central Rule hereunto annexed.

$$\text{Part}\begin{cases}1.\ \frac{1}{2}+\frac{p^2}{8}\underset{\backsim}{+}\frac{q}{2}=b=a\mathrm{D}\text{ or }\mathrm{AD}.\\ 2.\ \frac{p}{4}+\frac{p^2}{16}\underset{\backsim}{+}\frac{pq}{4}\underset{\backsim}{+}\frac{r}{2}=d=\mathrm{DH}.\end{cases}$$

To which Premonition of *Baker* we may alſo add this, if any given Line in the Problem itſelf be taken for Unity, which may be often very commodiouſly done, (as *a* in the former Problem, *p*. 91. *Geom. Cartes*, and the Line NO in the latter, and *a* again in the Equation, *p*. 83.) and then the ſame Line alſo may be taken for the *Latus Rectum* of the Parabola to be deſcribed, if we have a mind to make uſe of this Compendium for abbreviating the central Rules. For otherwiſe if we would conſtruct all Problems, as *Baker* rightly aſſerts we may, by only one Parabola, we ſhall fall often into very tedious Prolixities.

Some Examples of Specious Analyſis, in each kind of Equations.

I. *In Simple Equations.*

PROBLEM I.

THE Sums of every two Sides of a Triangle ABC, *being given, to find each of the Sides, and form the Triangle.*

Suppoſe, *e.g.* three Lines given in *Fig.* 14. the firſt $a=$ AB $+$ AC in the Triangle ſought, the ſecond $b=$ AB $+$ BC, and the third $c=$ BC $+$ AC, to find each of the Sides, *e.g.* to find AB, which being known, the reſt will be ſo alſo.

SOLUTION.

1. *Denomination.* Make AC $+$ AB $=a$; AB $+$ BC $=b$; BC $+$ AC $=c$; AB $=x$; then will AC $=a-x$, and BC $=b-x$, and ſo the Denomination will be compleat.

2. *Equation.* Now if the Values of the two laſt Lines BC and AC be added into one Sum, which we had before given; you'll have this Equation $a+b-2x=c$.

A SYNOPSIS of Mr. *Baker's* CLAVIS, to front *Page* 230, of the Introduction to *Specious Analyſis.*

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Of Equations.	*Of Equations.*	*Of Central Rules.*
Claſſ. I.	1. $x^4 *** - S = 0$	$\frac{s}{2} = b = AD,\ 0 = d = DH.$
II. 1. $x^2 * - q = 0$ 2. $x^2 * + q = 0$ *Impoſſible.*	$\begin{Bmatrix}1\\3\end{Bmatrix} x^4 * - qx^2 * \mp S = 0$ $\begin{Bmatrix}2\\4\end{Bmatrix} x^4 * + qx^2 * \mp S = 0$	$\left\{\begin{matrix}\frac{s}{2} + \frac{q}{2} \\ \frac{s}{2} \backsim \frac{q}{2}\end{matrix}\right\} = b = AD,\ 0 = d = DH$
III. 1. $x^3 ** - r = 0$ 2. $x^3 ** + r = 0$	$\begin{Bmatrix}1\\3\end{Bmatrix} x^4 ** - rx \mp S = 0$ $\begin{Bmatrix}2\\4\end{Bmatrix} x^4 ** + rx \mp S = 0$	$\frac{s}{2} = b = AD$ $\frac{r}{2} = d = DH$
1. $x^3 * - qx - r = 0$ 2. $x^3 * - qx + r = 0$	$\begin{Bmatrix}1\\3\end{Bmatrix} x^4 * - qx^2 - rx \mp S = 0$ $\begin{Bmatrix}2\\4\end{Bmatrix} x^4 * - qx^2 + rx \mp S = 0$	$\frac{s}{2} + \frac{q}{2} = b = AD$ $\frac{r}{2} = d = DH$
IV. 3. $x^3 * + qx - r = 0$ 4. $x^3 * + qx + r = 0$	$\begin{Bmatrix}5\\7\end{Bmatrix} x^4 * + qx^2 - rx \mp S = 0$ $\begin{Bmatrix}6\\8\end{Bmatrix} x^4 * + qx^2 + rx \mp S = 0$	$\frac{s}{2} \backsim \frac{q}{2} = b = AD$ $\frac{r}{2} = d = DH$
$\frac{p}{2} = BA$	$\frac{p}{2} = BA$	$\frac{p}{4} = AE$ *in the* 5, 6, 7, 8 *Claſs.*
V. 1. $x - p = 0$ 2. $x + p = 0$	$\begin{Bmatrix}1\\3\end{Bmatrix} x^4 - px^3 ** \mp S = 0$ $\begin{Bmatrix}2\\4\end{Bmatrix} x^4 + px^3 ** \mp S = 0$	$\frac{s}{2} + \frac{p^2}{8} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} = d = DH$
1. $x^3 - px^2 * + r = 0$ 2. $x^3 + px^2 * - r = 0$	$\begin{Bmatrix}1\\3\end{Bmatrix} x^4 - px^3 + rx \mp S = 0$ $\begin{Bmatrix}2\\4\end{Bmatrix} x^4 + px^3 * - rx \mp S = 0$	$\frac{s}{2} + \frac{p^2}{8} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} \backsim \frac{r}{2} = d = DH$
VI. 3. $x^3 - px^2 * - r = 0$ 4. $x^3 + px^2 * + r = 0$	$\begin{Bmatrix}5\\7\end{Bmatrix} x^4 - px^3 * - rx \mp S = 0$ $\begin{Bmatrix}6\\8\end{Bmatrix} x^4 + px^3 * + rx \mp S = 0$	$\frac{s}{2} + \frac{p^2}{8} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} + \frac{r}{2} = d = DH$
1. $x^2 - px - q = 0$ 2. $x^2 + px - q = 0$	$\begin{Bmatrix}1\\3\end{Bmatrix} x^4 - px^3 - qx^2 * \mp S = 0$ $\begin{Bmatrix}2\\4\end{Bmatrix} x^4 + px^3 - qx^2 * \mp S = 0$	$\frac{s}{2} + \frac{p^2}{8} + \frac{q}{2} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} = d = DH$
VII. 3. $x^2 - px + q = 0$ 4. $x^2 + px + q = 0$	$\begin{Bmatrix}5\\7\end{Bmatrix} x^4 - px^3 + qx^2 * \mp S = 0$ $\begin{Bmatrix}6\\8\end{Bmatrix} x^4 + px^3 + qx^2 * \mp S = 0$	$\frac{s}{2} + \frac{p^2}{8} \backsim \frac{q}{2} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} \backsim \frac{pq}{4} = d = DH$
1. $x^3 - px^2 - qx - r = 0$ 2. $x^3 + px^2 - qx + r = 0$	$\begin{Bmatrix}1\\3\end{Bmatrix} x^4 - px^3 - qx^2 - rx \mp S = 0$ $\begin{Bmatrix}2\\4\end{Bmatrix} x^4 + px^3 - qx^2 + rx \mp S = 0$	$\frac{s}{2} + \frac{p^2}{8} + \frac{q}{8} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} + \frac{r}{2} = d = DH$
3. $x^3 - px^2 - qx + r = 0$ 4. $x^3 + px^2 - qx - r = 0$	$\begin{Bmatrix}5\\7\end{Bmatrix} x^4 - px^3 - qx^2 + rx \mp S = 0$ $\begin{Bmatrix}6\\8\end{Bmatrix} x^4 + px^3 - qx^2 - rx \mp S = 0$	$\frac{s}{2} + \frac{p^2}{8} + \frac{q}{2} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} - \frac{r}{2} = d = DH$
VIII. 5. $x^3 - px^2 + qx + r = 0$ 6. $x^3 + px^2 + qx - r = 0$	$\begin{Bmatrix}9\\11\end{Bmatrix} x^4 - px^3 + qx^2 + rx \mp S = 0$ $\begin{Bmatrix}10\\12\end{Bmatrix} x^4 + px^3 + qx^2 - rx \mp S = 0$	$\frac{s}{2} + \frac{p^2}{8} \backsim \frac{q}{2} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} \backsim \frac{pq}{4} \backsim \frac{r}{2} = d = DH$
7. $x^3 - px^2 + qx - r = 0$ 8. $x^3 + px^2 + qx + r = 0$	$\begin{Bmatrix}13\\15\end{Bmatrix} x^4 - px^3 + qx^2 - rx \mp S = 0$ $\begin{Bmatrix}14\\16\end{Bmatrix} x^4 + px^3 + qx^2 + rx \mp S = 0$	$\frac{s}{2} + \frac{p^2}{8} \backsim \frac{q}{2} = b = AD$ $\frac{p}{4} + \frac{p^3}{16} \backsim \frac{pq}{4} + \frac{r}{2} = d = DH.$

N.B. *The Sign* $\backsim$ *denotes a dubious Caſe*, viz. *That either the Antecedent muſt be ſubſtracted from the Conſequent, or the Conſequent from the Antecedent, according as the Matter will bear.*

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3. *Reduction.* By adding on both ſides, $2x$, you'll have $a+b=c+2x$; and ſubſtracting from both ſides c ..... $a+b-c=2x$; and dividing both ſides by 2 ..... $\frac{a+b-c}{2}=x$.

4. The *Effection* or *Geometrical Conſtruction*, which the Equation thus reduc'd will help us to:

Join AE $=a$ and ED $=b$ in one Line AD, and from it cut off DF $=c$; and divide AF, which remains, into two equal parts in B, and you'll have AB the firſt ſide of the Triangle to be formed; and BE will give the other ſide AC, which ſubſtracted from ED, will leave GD $=$ to the third ſide BC; of which you may now form the Triangle ABC.

5. *A general Rule for Arithmetical Caſes.* Add the two former Sums, and from the Aggregate ſubſtract the third Sum; half the remainder will give the ſide AB common to the two former Sums. For an Example take this Queſtion; *There are three Towns of antient* Hetruria, *viz.* Forum Caſſii (*which the Letter* A *denotes in the* △ *ABC*) Sudertum (*B*) *and* Volſinii (*C*) *which are at this diſtance one from another*; *if you go from* Volſinii *to* Forum Caſſii, *and thence to* Sudertum, *you muſt go* 330 *Furlongs*; *from* Forum Caſſii *to* Sudertum, *and thence to* Volſinii, *there are* 306 *Furlongs*; *laſtly, from* Sudertum *to* Volſinii, *and thence to* Forum Caſſii, 272 *Furlongs. How far is each Town diſtant from the other?*

PROBLEM II.

IN *a right-angled Triangle* ABC, *having the Baſe* AB, *and the Difference of the Perpendicular* AC, *and the Hypothenuſe* BC *given, to find the Perpendicular and Hypothenuſe, and form the Triangle.*

Let *e. g.* the Baſe be AB, (*Fig.* 15.) and the difference of the Perpendicular and Hypothenuſe BD, to find the Perpendicular AC; which being known, the Hypothenuſe AC will be known alſo, if the given difference be added to the Perpendicular found.

SOLUTION.

1. *Denomination.* Make AB $=a$, BD $=b$, AC $=x$; then will BC $=x+b$.

2. *Equation* by the Pythagorick Theorem, $xx+aa=xx+2bx+bb$, viz. the two Squares of the Sides to the Square of the Hypothenuſe.

3. *Reduction.* Subſtracting from both ſides xx, you'll have $aa = 2bx + bb$; and moreover by ſubſtracting alſo bb, $aa - bb = 2bx$; and dividing by $2b$,

$$\frac{aa - bb}{2b} = x.$$

4. *Effection* or *Geometrical Conſtruction.* Having deſcribed a Semicircle upon the given Baſe AB, apply therein the given difference BD, and draw AD, whoſe Square is $= aa - bb$. Since this muſt be divided by $2b$, make, as AE $= 2b$ to AD $= \sqrt{aa - bb}$ ſo is AD $= \sqrt{aa - bb}$ to AC the Perpendicular ſought. To which if you add CF = BD, you will have AF = to the Hypothenuſe ſought BC; which will come of courſe together with the whole Triangle ſought, if the found Perpendicular AC be erected at right Angles on the given Baſe AB.

5. *The Rule for Arithmetical Caſes.* From the Square of the given Baſe ſubſtract the Square of the given Difference, and divide the Remainder by the double difference; and you'll have the Perpendicular ſought. *E. g.* ſuppoſe the Baſe = 20 foot, and the Difference between the Perpendicular and Hypothenuſe 10.

PROBLEM III.

IN the right-angled Triangle ABC, *having given the Side* AC, *and the Sum of the other Side* AB, *and Hypothenuſe* BC; *to find the other Side and the Hypothenuſe ſeparately, and form the Triangle.* Suppoſe the given Side (that is to be) AC, (*Fig. 16.*) and the Sum of the other two AD; to find the Side AB, which being known, the Hypothenuſe BC will be known alſo.

SOLUTION.

1. *Denomination.* Make AC $= a$, AD $= b$, AB $= x$, then will BC $= b - x$.

2. *Equation.* $xx + aa = bb - 2bx + xx$, and ſubſtracting xx.

3. *Reduction.* $aa = bb - 2bx$; and adding $2bx$,
$aa + 2bx = bb$; and ſubſtracting aa,
$2bx = bb - aa$; and dividing by $2b$,

$$x = \frac{bb - aa}{2b}.$$

4. The *Effection* or *Conſtruction* is like the former, and ſo will be manifeſt only by inſpecting *Fig. 16.*

5. *The Arithmetical Rule.* From the Square of the given Sum ſubſtract the Square of the given Side, and divide the Remainder by double the given Sum; and you will have the other Side, and ſubſtracting that from the given Sum, you have the Hypothenuſe alſo. *E. g.* let one Side be 15, and the Sum of the other two 45.

PROBLEM IV.

THE Perpendiculars and Sum of the Baſes of two right-angled Triangles, having equal Hypothenuſes, being given, to find the Baſes ſeparately, and form the Triangles. Suppoſe, *e.g.* to form the Triangle ABC (ſee *Fig.* 17.) you have given the Perpendicular AB; and for the other △ ADC, the Perpendicular CD, and let the given Sum of the Baſes be BE, to find the Baſes ſingly, *viz.* the leſs BC for the greateſt Perpendicular, and the greater AD for the leſs Perpendicular.

SOLUTION.

1. *Denomination.* Make $AB = a$, $CD = b$, the Sum $BE = c$; make the leſſer Baſe $BC = x$; then the greater AD will $= c - x$.

2. *Equation.* Since the Hypothenuſes of the two Triangles are ſuppoſed equal, the two □ □ of AB + BC, *i.e.* $xx + aa$ will be = to the two □ □ of AD + CD, *i.e.* $bb + cc - 2cx + xx$.

3. *Reduction.* Therefore by taking away xx, and adding $2cx$, $aa + 2cx$ will $= bb + cc$; and further taking away from both Sides, aa, $2cx = bb + cc - aa$; and dividing both Sides by $2c$, $x = \dfrac{bb + cc - aa}{2c}$.

4. *Geometrical Conſtruction.* Join the Lines b and c, *i. e.* CD and BE at right Angles, (*n.* 2.) and the Square of the Hypothenuſe DE will $= bb + cc$. Upon this Hypothenuſe having deſcribed a Semi-circle, apply therein the Line AD, and the Square of AE will $= bb + cc - aa$. Which, ſince it muſt be further divided by $2c$, make (*n.* 3.) as $BF = 2c$, to BG, $= \sqrt{bb + cc - aa}$, ſo is BG to BC, the leſſer Baſe ſought.

5. *The Arithmetical Rule.* From the Sum of the Squares of the leſſer Perpendicular and the Sum of the Baſes, ſubſtract the Square of the greater Perpendicular, and the Remainder divided by double the Sum of the Baſes, will give the leſſer Baſe. *E. g.* let AB be 76, CD 57, and BE 114.

PRO-

PROBLEM V.

HAving given the Perpendiculars of two right-angled Triangles ſtanding contrary ways on the ſame given Baſe; to find the Segments of the Hypothenuſes. E.g. ſuppoſe the common given Baſe be AB (*Fig.* 18.) and the Perpendicular of one Triangle AD, and of the other BC; to find geometrically the Segments of the Hypothenuſes, cutting one another.

SOLUTION.

If a Geometrical Solution be required, there is no need of any Analyſis; for having erected the given Perpendiculars AD, BC, perpendicularly on the common given Baſe AB, the Hypothenuſes AC, BD, being drawn, their Segments EA, EB, EC, ED, will be determin'd without any more to do. But if it be to be done arithmetically by a general Rule, then an Analyſis will be neceſſary.

1. *Denomination.* Make the common Baſe AB $= a$, BC $= b$, AD $= c$; and, having found AF, all the reſt may be had (for as AB to BC, ſo is AF to FE; which being given, you have alſo GD and HC, and conſequently alſo DE, CE, *&c.*) make AF $= x$; then will BF or HE $= a - x$.

2. The *Equation* from FE found twice.

(1.) As AB to BC, ſo is AF to FE.

$$a \ldots\ldots b \ldots\ldots x \ldots\ldots \frac{bx}{a}.$$

(2.) As BA to AD, ſo is BF to FE.

$$a \ldots\ldots c \ldots\ldots a - x \ldots\ldots \frac{ac - cx}{a}.$$

Therefore $\frac{bx}{a} = \frac{ac - cx}{a}$.

3. *Reduction.* Multiplying both Sides by a, you'll have $bx = ac - cx$; and adding on both Sides cx, $bx + cx = ac$; and dividing both Sides by $b + c$, $x = \frac{ac}{b + c}$.

4. *The Arithmetical Rule.* Multiply the common Baſe by the leaſt Perpendicular, and divide the Product by the Sum of the Perpendiculars; and you'll have the leſſer Segment of the Baſe, which being given, you'll have all the reſt. *E.g.* ſuppoſe AB $= 10$, BC $= 9$, AD $= 6$.

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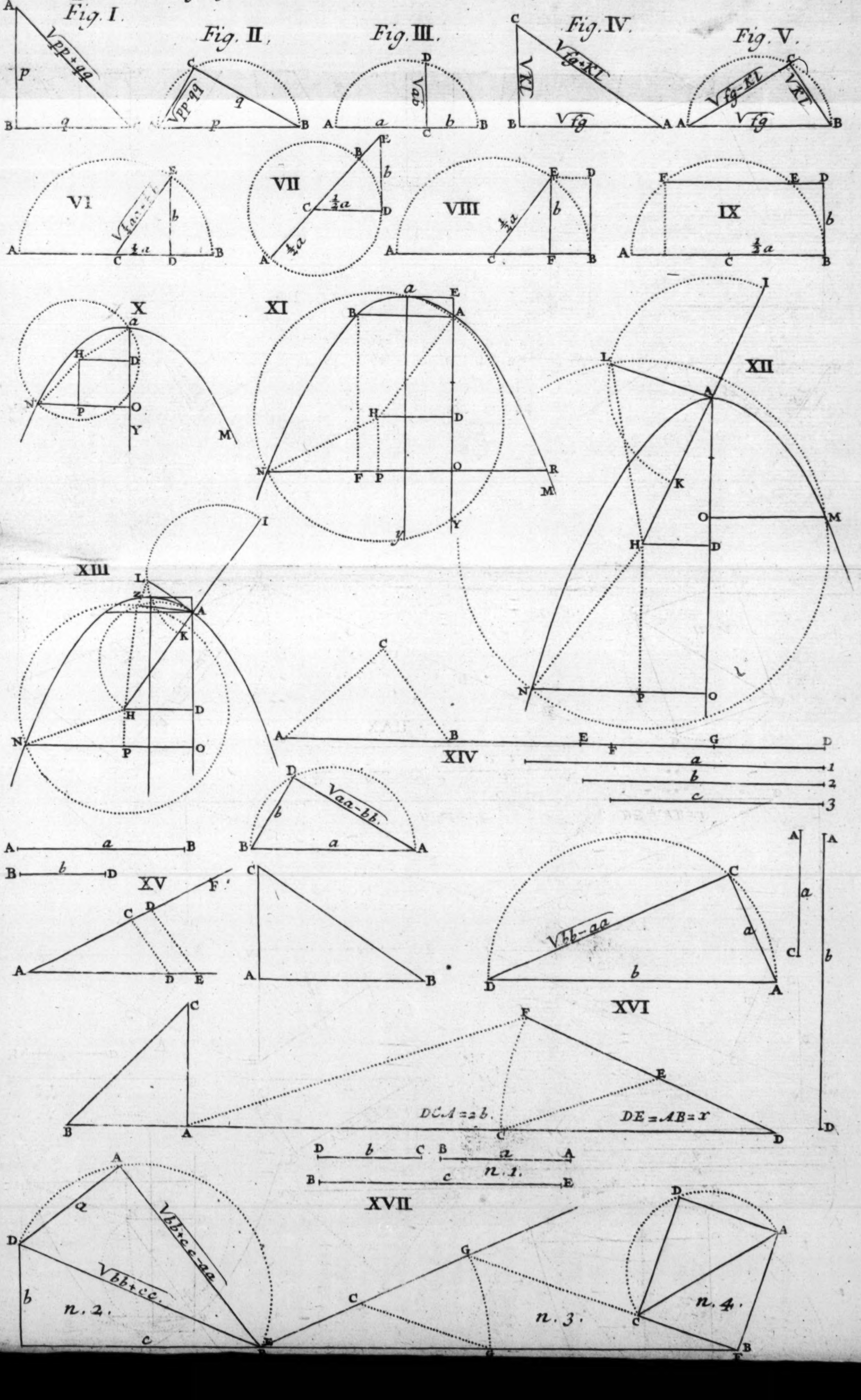

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PROBLEM VI.

TO inſcribe a Rhombus in a given Oblong, i. e. *the Sides of the Oblong* AB *and* BC *being given,* (*Fig.* 19.) *to find the Segment* BF *or* DE, *which being cut off, the Remainder FC or AE may be the Side of the Rhombus ſought.*

SOLUTION.

Make $AB = a$, $BC = b$, $BF = x$: then FC, or FA will be $= b - x$ (ſo far the *Denomination.*) Therefore the Square of FA, which is $bb - 2bx + xx$ will be $= aa + xx$, *viz.* to the two Squares of AB and BF (ſo far the *Equation*;) and ſubſtracting from both ſides, xx, $bb - 2bx = aa$; and
adding $2bx$ on both ſides, $bb = aa + 2bx$, and
ſubſtracting aa, $bb - aa = 2bx$; and
dividing by $2b$, $\frac{bb - aa}{2b} = x$ (ſo far the *Reduction.*)

The Geometrical Conſtruction. Having deſcribed a Semicircle on BC (*n.* 2.) apply CD or AB in it, and the □ of BD will be $= bb - aa$. Which ſince it muſt be divided by $2b$, make, as $BE = 2b$, to $BD = \sqrt{bb - aa}$, ſo is BD to BF ſought, which is to be cut off from the ſide of the Oblong BC, (*n.* 1.)

The Arithmetical Rule. From the Square of the greater Side ſubſtract the Square of the leſſer, and divide the remainder by double the greater Side; and the Quotient will give the Segment BF ſought. *E. g.* ſuppoſe $AB = 4$, and $BC = 8$.

PROBLEM VII.

TO inſcribe the greateſt Square poſſible in a given Triangle, i. e. having given the Height of the Triangle CD (*Fig.* 20.) and the Baſe AB, to find a Portion of the Altitude CE, which being cut off, there ſhall remain ED = FG.

SOLUTION.

Make the Baſe $AB = a$, the Altitude $CD = b$, and $CE = x$; then will ED or $FG = b - x$.

By reaſon of the ſimilitude of the Triangles ABC and FGC, it will be as AB to CD, ſo is FG to CE.

$$a : b :: b - x : x.$$

Therefore the Rectangles of the Means and Extremes will be equal, *i. e.* $ax = bb - bx$; and adding on both ſides bx, $ax + bx = bb$, and dividing by $a + b$, $x = \frac{bb}{a+b}$.

Conſtruction. In the Side of the Triangle CB produced, take CH$=b$, and HI$=a$, ſo that the whole Line be $a+b$. And having joined ID, and from H drawn HE parallel to it, the Part CE will be cut off, which is that ſought. For as CI to CD, ſo is CH to CE,

$a+b : b :: b : x$, according to the ſecond Caſe of ſimple Effections.

Arithmetical Rule. Square the given Height of the Triangle, and divide the Product by the Sum of the Baſe and Altitude; and the Quotient is the Part to be cut off CE. *E. g.* ſuppoſe CD$=10$, and AB$=15$.

PROBLEM VIII.

IN an acute-angled Triangle, having all the Sides given, to find the Perpendicular falling from the Vertex on the Baſe, i. e. having AB, AC, BC (*Fig.* 21.) given, to find AD or BD; (for having found the one, you may eaſily find the other.) *Coroll. Prop.* 13. *lib.* 2. *Eucl.*

SOLUTION.

If only a geometrical Conſtruction of this Problem be required, there will be no need of any Analyſis; for having formed a Triangle ABC of the three given Sides, you need only let fall the Perpendicular AD from the Vertex A, which will determine the Segment BD. But to find the general arithmetical Rule, which is the general Corollary of *Euclid*; or if any one, for Exerciſe ſake, had rather determine the Perpendicular DA by the Segment of the Baſe BD, than the latter by the former, the Analyſis will be thus:

Make AB$=a$, BC$=b$, AC$=c$, BD$=x$; then will CD be$=b-x$. Wherefore, by the *Pythagorick Theorem*, the □ of AD$=aa-xx$; and, by the ſame reaſon, the ſame □ of AD$=cc-bb+2bx-xx$. Therefore $aa-xx=cc-bb+2bx-xx$; and by adding on both ſides xx, $aa=cc-bb+2bx$; and by tranſferring $cc-bb$, $aa-cc+bb=2bx$, and dividing by $2b$, $\frac{aa-cc}{2b}+bb=x$.

The Arithmetical Rule. Subſtract the Square of the leſſer Side from the Sum of the Squares of the Baſe and greater Side, and the Remainder divided by double the Baſe, will give its greater Segment: If the □ of the greateſt Side be ſubſtracted from the Sum of the Squares of the others, *&c.* you will have the leſs Segment CD.

Geometrical Conſtruction. Having deſcribed a Semicircle upon AB (*N.* 2.) apply therein AC, and the □ of BC will $= aa - cc$; and continuing AC to 𝔅, till C𝔅 be = CB (*N.* 1.) the □ of 𝔅B will $= aa - cc + bb$; which, ſince it is to be divided by $2b$, make as 𝔅E $= 2b$, to 𝔅B $= \sqrt{aa - cc + bb}$, ſo is 𝔅F = 𝔅B, to 𝔅D the Segment ſought = BD. *N.* 1.

PROBLEM IX.

IN an obtuſe-angled Triangle, having the three Sides given, to find the Perpendicular let fall from the Vertex to the Baſe being continued: i. e. Having given AB, BC, AC (*Fig.* 22. *N.* 1.) to find AD or CD (for the one being found, the other will readily be ſo alſo.) *Coroll. Prop.* 12. *lib.* 2. *Eucl.*

SOLUTION.

What we premoniſhed about the former Problem, we underſtand to be premoniſhed here alſo. For the reſt make here alſo AB $= a$, BC $= b$, AC $= c$, CD $= x$; then will BD $= b + x$: Wherefore, by the *Pythagorick Theorem*, the □ of AD will $= cc - xx$, and by the ſame *Theorem*, the ſame □ of AD $= aa - bb - 2bx - xx$.

Therefore $cc - xx = aa - bb - 2bx - xx$; and adding xx to both Sides, $cc = aa - bb - 2bx$; and tranſpoſing cc and $-2bx$, $2bx = aa - cc - bb$; and dividing by $2b$, $x = \frac{aa - cc - bb}{2b}$.

The Arithmetical Rule. From the Square of the greater Side, ſubſtract the Sum of the Squares of the Baſe and leſſer Side; and the Remainder divided by double the Baſe, will give its continuation to the Perpendicular.

Geometrical Conſtruction. From the Equation reduced: having deſcribed a Semicircle upon AB (*N.* 2.) apply therein AC; and on CB deſcribe another Semicircle, wherein apply BE $= b$, and the □ of CB being drawn, will be $= aa - cc - bb$; which, ſince it muſt be divided by $2b$, make as CF $= 2b$, to CE $= \sqrt{aa - cc - bb}$, ſo is CE to CD the Segment ſought. *N.* 1.

PROBLEM X.

Commonly aſcribed to Archimedes.

THE Diameter AB *of a given Semicircle* (Fig. 23. N. 1.) *being any how divided in* L, *and from* L *erecting a Perpendicular* LX, *and upon the Segments* LA *and* LB, *having deſcribed two other Semicircles, whoſe Semidiameters are alſo given, as well as* AB, *that of the greater Semicircle* CB; *to find the Radii* FM *and* VY *of the little Circles that are to be ſo deſcribed, that they ſhall touch the Perpendicular* LX, *the Cavity of the greater Semicircle, and the Convexities of the two leſſer ones.*

SOLUTION.

I. *For the Radius* FM.

1. *Denomination.* Make CB $=a$, EB $=b$; then will CE $=a-b$; for which (for brevity's ſake) put c. And let FM, or FN, or FK $=x$: Therefore EF will be $=b+x$, and CF (ſubſtracting FK from CK) $=a-x$. Wherefore now you'll have at leaſt the Names of the three Sides in the △ CFE, ſo that according to Problem 8. the Segment of the Baſe GE may be determined (which indeed is determined already, as being $=$ LE — LG or MF, *i. e.* $b-x$) for which, in the mean time, we will put y; and now will CG $=c-y$.

2. For the *Equation.* If the □ of GE $=yy$ be ſubſtracted from the □ of EF $=bb+2bx+xx$, you'll have the Square of the Perpendicular FG $=bb+2bx+xx-yy$; and, if the □ of CG $=cc-2cy+yy$ be ſubſtracted from the □ of CF $=aa-2ax+xx$, you'll have the ſame □ of the Perpendicular FG $=aa-2ax+xx-cc+2cy-yy$. Therefore $bb+2bx+xx-yy=aa-2ax+xx-cc+2cy-yy$.

3. *Reduction.* And taking from both ſides the Quantities xx and yy, then will $bb+2bx=aa-2ax-cc+2cy$; and adding $2ax$ and cc, and taking away from each ſide aa, $bb+2bx+2ax+cc-aa=2cy$; and dividing by $2c$, $\frac{bb+2bx+2ax+cc-aa}{2c}=y$; but the ſame y or EG is $=$ EL — MF, *i.e.* $b-x$. Therefore $\frac{bb+2bx+2ax+cc-aa}{2c}=b-x$; which is a new and more principal Equation:

And multiplying both ſides by $2c$ (to have a new Reduction) $bb+2bx+2ax+cc-aa=2bc-2cx$; and adding $2cx$, and tranſpoſing $2bx+2cx+2ax=aa-bb-cc+2bc$; and dividing by $2a+2b+2c$, $x=\frac{aa-bb-cc+2bc}{2a+2b+2c}$.

The Geometrical Conſtruction of this Caſe. Add the determinate (*N.* 2.) Quantities $2bc$, aa, into one Sum, as in *N.* 3. Then from this Sum ſubſtract ſucceſſively the Quantities bb and cc, and there will come out FH, (*N.* 3.) whoſe $\square=aa+2bc-bb-cc$: Which, ſince it muſt be divided by $2a+2b+2c$, make (*N.* 3.) as $\mathrm{FI}=2a+2b+2c$, to $\mathrm{FH}=\sqrt{aa+2bc-bb-cc}$, ſo is FH to FM the Radius ſought of the little Circle to be deſcribed. This Quantity FM being thus found, place it from L to G, (*N.* 1.) and from G erect a Perpendicular; then, if on the Center C, with the Diſtance CF, which will be had by taking $\mathrm{FK}=\mathrm{FM}$ from CB or CK, or on the Center E, with the Diſtance $\mathrm{EF}=\mathrm{EN}+\mathrm{FN}$, you deſcribe an Arc cutting the Perpendicular in F: this will be the Center of the little Circle ſought.

The Arithmetical Rule. Add twice the ▭ CEB to the Square of the greateſt Semidiameter CB, and from the Sum ſubſtract the Aggregate of the Squares of CE and EB; divide the Remainder by the Sum of all the three Diameters (AB, AL, and LB) *i. e.* by double the greateſt AB; and you'll have the Radius FM, &c. For example, let a be $=12$, $b=4$; then c will be $=8$, and x will be $=2\frac{2}{3}$.

II. *For the Radius* VY, *by help of the obtuſe-angled Triangle* DVC.

1. *Denomination.* Let CA be $=a$, as above, DA or DL $=b$, and putting x again for the ſought Radius VY or VK; CV will be $=a-x$, DL or DR $=b$, and conſequently DV $=b+x$, and DC $=a-b$, for which (for brevity's ſake) we will put c. Now you'll have at leaſt in Denomination the three Sides of the $\triangle$ CVD, ſo that according to Problem 9. the Segment CW may be determined; for which, in the mean while, we will put y: then will DW $=c+y$, which is the ſame as DL — WL, or VY, *i. e.* $b-x$.

2. For the *Equation.* If the $\square$ of CW $=yy$ be ſubſtracted from the $\square$ of CV $=aa-2ax+xx$, you'll have the $\square$ of the Perpendicular VW $=aa-2ax+xx-yy$; and if the $\square$ of DW $=cc+2cy+yy$ be ſubſtracted from the $\square$ of

$DV = bb + 2bx + xx$, you'll have the ſame □ of the Perpendicular $VW = bb + 2bx + xx - cc - 2cy - yy$. Therefore $aa - 2ax + xx - yy = bb + 2bx + xx - cc - 2cy - yy$.

3. *Reduction.* Taking xx, yy from both ſides, and then $aa - 2ax = bb + 2bx - cc - 2cy$; and adding $2cy$ and $2ax$, $aa + 2cy = bb + 2ax + 2bx - cc$; and ſubſtracting aa, $2cy = bb - aa + 2ax + 2bx - cc$, and dividing by $2c$, $y = \frac{bb - aa + 2ax + 2bx - cc}{2c}$: But if to the ſame y or CW you add $DC = c$, you'll have $\frac{bb - aa + 2ax + 2bx + cc}{2c} = DW$. But the ſame $DW = DL - WL = b - x$. Therefore $\frac{bb - aa + 2ax + 2bx + cc}{2c} = b - x$, and multiplying by $2c$, $bb - aa + 2ax + 2bx + cc = 2bc + 2cx$, and adding $2cx$, and tranſpoſing the reſt, $2ax + 2bx + 2cx = aa - bb - cc + 2bc$, and dividing by $2a + 2b + 2c$, $x = \frac{aa - bb - cc + 2bc}{2a + 2b + 2c}$, juſt as above in the firſt Caſe.

4. *The Geometrical Conſtruction* therefore will be the ſame as there. See *Fig.* 23. *N.* 4, and 5.

5. *The Arithmetical Rule* is alſo the ſame; but the given Quantities in this Example, which the Figure of the Problem will ſhew, thus vary: while a remains 12, b will be 8, and c, 4; from which *Data* (or given Quantities) there will, notwithſtanding, come out again $2\frac{2}{3}$ for x or the Radius VY.

II. *Some Examples of ſimple or pure quadratick Equations.*

PROBLEM I.

TO make a Square equal to a given Rectangle, i. e. *the Sides of a Rectangle being given, to find the Side of a Square equal to it:* Eucl. Prop. 14. lib. 2. Suppoſe, *e. g.* the given Sides of the Oblong to be AB and BC (*Fig.* 24.) to find the Line BD, whoſe Square ſhall be equal to that Rectangle.

SOLUTION.

Make $AB = a$, $BC = b$, and the Side of the Square ſought $= x$, then the Equation will be $ab = xx$; and extracting the Root on both Sides $\sqrt{ab} = x$.

Geometrical Conſtruction. Join AB and BC in one right Line, and deſcribing a Semicircle upon the whole AC, from the common Point B erect the Perpendicular BD; which will be the Side of the Square ſought, according to Caſe 1. of the Conſtruction of pure Quadraticks.

Arithmetical Rule. Multiply the given Sides of the Oblong by one another, and the ſquare Root of the Product being extracted, will be the Side of the Square ſought.

PROBLEM II.

THE *Square of the Hypothenuſe in a right-angled* △ *being given, as alſo the Difference of the Squares of the other two Sides; to find the Sides:* that is, if the Hypothenuſe be BC (*Fig.* 25.) and the Difference of the Squares of both the Legs, and conſequently its Side BE be alſo given, (for the Squares being given, the Sides are alſo given geometrically) it is required to find the Sides of the right-angled △ which ſhall have theſe Conditions: or more plainly, to find one Side, *e. g.* the leſſer AC; which being found, the other, or the greater Side, will be found alſo.

SOLUTION.

Let the □ of the given Hypothenuſe be $= aa$, and the Difference of the Squares of the Legs $= bb$. Let the leſs Side $= x$, then its □ will be $= xx$. Wherefore the Square of the greater will be $xx + bb$. And ſince the Sum of theſe is = to the □ of the Hypothenuſe, you'll have

$2xx + bb = aa$; and ſubſtracting bb,

$2xx = aa - bb$; and dividing by 2

$xx = \frac{aa - bb}{2}$. Therefore

$$x = \frac{\sqrt{aa - bb}}{2}.$$

Geometrical Conſtruction. Having deſcribed a Semicircle on BC, and applied therein BE, the □ of EC will be $= aa - bb$; and having deſcribed another Semicircle upon EC divided into two Quadrants, the □ of DC will be $= \frac{aa - bb}{2}$, and ſo DC $= \frac{\sqrt{aa - bb}}{2}$, or the Side ſought; which being alſo transferred upon another Semicircle deſcrib'd on BC, from C to A, gives the other Side AB, and ſo the whole △ ſought.

The Arithmetical Rule. From the Square of the Hypothenuſe, ſubſtract the given Difference; and the ſquare Root of half the Remainder being extracted, gives the leſſer Side of the △ ſought.

PROBLEM III.

HAving an equilateral △ ABC *given* (Fig. 26. N. 1.) *to find the Center and Semidiameter of a Circle that ſhall circumſcribe it*; i.e. to find BD the Side of an Hexagon that may be inſcribed in the ſame Circle. For if we conſider the thing as already done, it is manifeſt, that BD the Side of the Hexagon will fall perpendicularly on the Side of the △ AB, as making an Angle in a Semicircle, ſo that having biſected the Hypothenuſe DA, you'll have E the Center ſought.

SOLUTION.

Let the Side AB of the Triangle be $= a$, and let BD $= x$; then will AD $= 2x$. Since therefore the Square of BD, *i.e.* xx, being ſubſtracted out of the Square of AD, *i.e.* $4xx$, there remains the Square of AB $= 3xx$, you'll have the Equation

$$3xx = aa; \text{ and dividing by } 3$$

$$xx = \frac{aa}{3}; \text{ therefore}$$

$$x = \frac{\sqrt{aa}}{3}.$$

The Geometrical Construction. Having produced AB (*N.* 2.) to F, ſo that BF be a third Part of it; the Square of a mean Proportional BD between BF and BA, will be $\frac{1}{3}aa$ or $\frac{aa}{3}$; and ſo the Line BD $= \frac{\sqrt{aa}}{3}$. Therefore the Hypothenuſe DA being biſected in E, or elſe with the Interval BD, making an Interſection from B and A, and you'll have the Center ſought.

The Arithmetical Rule. Divide the Square of the given Side into three equal Parts, and the ſquare Root of a third Part will give the Semidiameter AE or BE ſought, by the Interſection whereof you may have the Center.

PRO-

PROBLEM IV.

THE Diagonal of a right-angled Parallelogram, and the Proportion of the Sides, or the Hypothenuſe, and the Proportion of the Sides of a right-angled Triangle being given: to find the Sides ſeparately, and conſtruct the Parallelogram or Triangle. Suppoſe, *e.g.* the given Diagonal to be AB, (*Fig.* 27. *N.* 1.) and the given Ratio of the Sides to be as AD to DE: to find the Sides.

SOLUTION.

Make AB $= a$, the Ratio of AD to DE, as b to c; make the leſſer Side $= x$; and then will the greater be $\frac{cx}{b}$.

For the *Equation*. The Squares of the Sides are $xx + \frac{ccxx}{bb} = aa$, the □ of AB; and multiplying both Sides by bb, $bbxx + ccxx = aabb$; and dividing by $bb + cc$, $xx = \frac{aabb}{bb+cc}$. Therefore $x = \sqrt{\frac{aabb}{bb+cc}}$, *i. e.* extracting the Roots as far as poſſible, $x = \frac{ab}{\sqrt{bb+cc}}$.

Another Solution.

Put e for the Name of the given Ratio; ſo that aſſuming any Line, as AD, for Unity, the Value of e may alſo be expreſſed by a right Line, which ſhall be equal, *e. g.* to DE above. Wherefore, becauſe we make the leſs Side x, the greater will be ex; and ſo $xx + eexx = aa$, *i. e.* dividing by $1 + ee$, $xx = \frac{aa}{1+ee}$, and $x = \sqrt{\frac{aa}{1+ee}}$, i. e. $x = \frac{a}{\sqrt{1+ee}}$.

The Geometrical Conſtruction. The laſt Equation above being reduced to this Proportion, as $\sqrt{bb+cc}$ to b, ſo is a to x, make (*N.* 2.) AD and DE at right Angles, and AE will be $= \sqrt{bb+cc}$; and, continuing AE and AD, make as AE to AD, ſo is AB to AC the leſſer Side ſought. Therefore having drawn BC, which determines the leſſer Side AC; the greater Side, and ſo the △ ABC will be already formed, and may be eaſily compleated into a Rectangle.

angle. In the other Solution, the laſt Equation agrees with the precedent one; for it gives us this Proportion, as $\sqrt{1+ee}$ to 1, ſo is a to x, in which 1 is $=b$, and $ee=cc$ (by what we have ſuppoſed;) and ſo the Conſtruction will be the ſame.

The Arithmetical Rule may be more commodiouſly expreſſed by this laſt Equation, under the laſt Form but one, after this way: Divide the □ of the Diagonal by the □ of the Name of the Ratio augmented by Unity; and the Root extracted out of the Remainder, is the leſſer Side ſought.

PROBLEM V.

(Which is in *Pappus Alexandrinus*, and in *Des Cartes*'s Geometry, *p.* 83. in a biquadratick affected Equation; and, *p.* 84. he gives us thereon a very remarkable Note.)

THE *Square* AD, (Fig. 28.) *and a right Line* BN *being given, to produce the Side* AC *to* E, *ſo that* EF *drawn from* E *to* B, *ſhall be equal to* BN.

If you imagine a Semicircle to paſs thro' the Points B and E, it will be evident, that the moſt commodious way will be to find the Line DG, that ſo you may have the Diameter BG; upon which having afterwards deſcribed a Semicircle, there will be need of no other Operation to ſatisfy the Queſtion, but to produce the Side AC till it meets the preſcribed Periphery.

SOLUTION.

(*As found by* Van Schooten, *p.* 316. *in his Comment on* Des Cartes's *Geometry, which we will here give ſomewhat more diſtinct.)*

1. *Denomination.* Make BD or DC $=a$, BN or FE $=c$, BF $=y$, and DG $=x$; then the Perpendicular EH will be $=a$, and EG = BF, *i. e.* y (becauſe the △ EHG is ſimilar to △ BDF, by *N.* 3. *Schol.* 2. *Prop.* 34. *Lib.* 1. *Math. Enucl.* and BD in the one = to EH in the other) and BG $=a+x$, BE $=y+c$; and BH will have its Denomination, if you make (by reaſon of the Similarity of the Triangles BFD and BEH)

$$\begin{array}{ccccccc} BF & : & BD & :: & BE & : & BH \\ y & : & a & :: & y+c & : & \dfrac{ay+ac}{y} \end{array}$$

; then you'll have alſo HG

=

$= a + x - \frac{ay - ac}{y}$, *i.e.* having reduced them all to the ſame Denominator $\frac{ay + xy - ay - ac}{y}$, or $\frac{xy - ac}{y}$. Therefore having named all the Lines you have occaſion for, you muſt find two Equations, becauſe there are aſſumed two unknown Quantities, *viz.* x and y.

2. *For the firſt Equation and its Reduction.* By reaſon of the Similarity of the Triangles BGE and BEH,

BG : GE :: BE : EH

$a + x : y :: y + c : a$: Therefore the Rectangle of the Extremes will be = to the Rectangle of the Means, i. e. $yy + yc = aa + ax$; and taking from both Sides yc, $yy = aa + ax - yc$.

3. *For the ſecond Equation and its Reduction.* Since BH, HE, and HG, that is, $\frac{ay + ac}{y}$, a, $\frac{xy - ac}{y}$, are continual Proportionals, the ▭ of the Extremes is equal to the Square of the Mean, *i.e.* $\frac{axyy + acxy - aacy - aacc}{yy} = aa$; and multiplying both Sides by yy, and dividing by a, $xyy + cxy - acy - acc = ayy$, and taking away ayy, and tranſpoſing the reſt, $xyy - ayy = acy - cxy + acc$; and dividing by $x - a$, $yy = \frac{acy - cxy + acc}{x - a}$; *i.e.* dividing actually as far as may be by $x - a$, and then $yy = y - cy + \frac{acc}{x - a}$.

4. *The Compariſon of theſe two Equations thus reduced, gives a third new one, in which there will be only one unknown Quantity,* viz. $-cy + aa + ax = -cy + \frac{acc}{x - a}$; and adding to both Sides cy, $aa + ax = \frac{acc}{x - a}$; and multiplying by $x - a$, $aax + axx - a^3 - aax = acc$; i. e. $axx - a^3 = acc$; and dividing both Sides by a, $xx - aa = cc$; and adding aa, $xx = aa + cc$. Therefore $x = \sqrt{aa + cc}$.

5. *The Geometrical Conſtruction*, which is the ſame *Pappus* preſcribes, as *Des Cartes* has it, *viz.* having prolong'd the Side of the Square BA to N, ſo that BN be = to the given

right Line, ſince BA is $= a$, and BN $= c$; the Hypothenuſe DN will be $= \sqrt{aa + cc} = x$. Having therefore made DG $=$ DN, and deſcribed a Semicircle upon the whole Line BG; if AC be prolonged until it meets the Periphery in E, you'll have done that which was required.

PROBLEM VI.

(Which *Van Schooten* has in his Comment, *p. m.* 150. and following.)

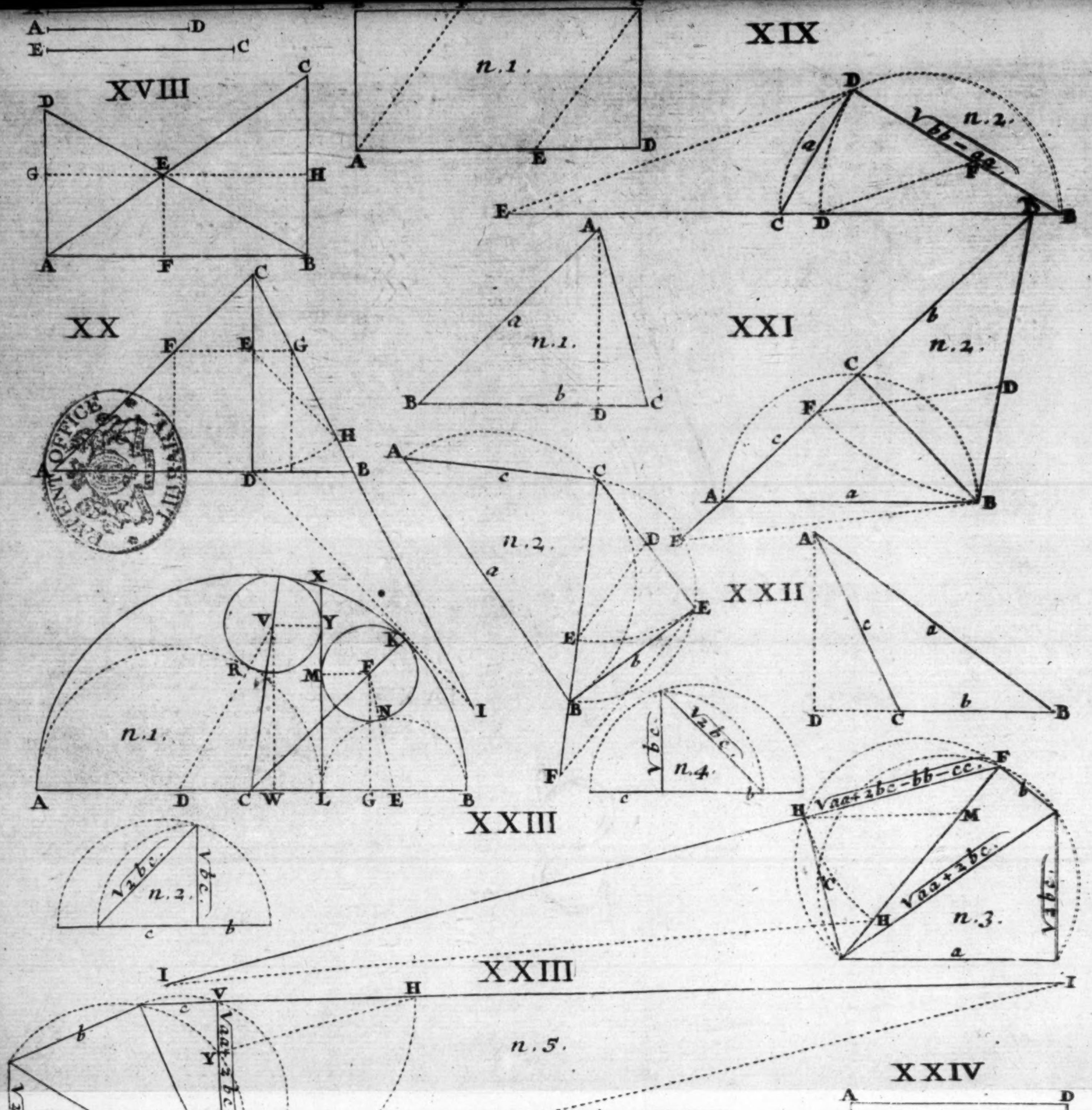
A Finite right Line AB *being given* (Fig. 29.) *to draw two right Lines* AC *and* BC, *from its Extremes* A, B, *which ſhall contain an Angle* ACB $=$ *to the given one* D, *and whoſe Squares ſhall be in a given Proportion to the Triangle* ACB, viz. *as* $4d$ *to* a.

You muſt determine the Point C, which the two right Lines AH and HC, or EH and HC will do, aſſuming the Point E, being the middle of the Line AB. Wherefore here will be two unknown Quantities HE and HC, and conſequently two Equations to be found in the Solution; one whereof the given Proportion in the Queſtion ſupplies us with; and the other, we have from the ſimilar Triangles AIC and GFD, which repreſent equal Angles.

SOLUTION.

Denomination. Make AE the half of AB $= a$, HE $= x$ and HC $= y$; then AH will be $= a - x$, and HB $= a + x$; whence the Denomination of the Squares of AC and BC is eaſily had; *viz.* the one $aa - 2ax + xx + yy$; and the other, $aa + 2ax + xx + yy$, ſo that the Sum of the Squares is $2aa + 2xx + 2yy$. And the $\triangle$ ACB will be $= ay$: And ſince the Triangles GFD and AIC are ſimilar, and the Sides of the former FD and FG may be taken at pleaſure; ſo that for FD, we may put b; and for FC, c; and the Sides of the latter AI, IC are determined by the Similitude of the Triangles ABI and HCB, as being right-angled ones, and having the common Angle B; they will be obtained by making as the Hypothen. BC to the Hypothen. AB; ſo is the Perpendicular CH to the Perpendicular AI: that is, as $\sqrt{aa + 2ax + xx + yy}$, which let be $= e$, to $2a$, ſo is y to $\frac{2ay}{e}$.

And



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And as the Hyp. BC to the Hyp. AB, ſo is the Baſe HB to the Baſe BI; that is, $e : 2a :: a+x : \frac{2aa+2xx}{e}$. from whence ſubſtracting BC $=e$, and there remains CI $= 2aa + \frac{2ax-ee}{e}$.

2. *For the firſt Equation.* By virtue of the Problem, as $4d$ to a, ſo is $2aa+2xx+2yy$ to ay. Therefore the Rectangle of the Extremes is $=$ to the Rectangle of the Means, *i. e.* $4ady = 2a^3 + 2axx + 2ayy$.

3. *For the other Equation.* Since DF : FG :: CI : AI, $b : c :: \frac{2aa+2ax-ee}{e} : \frac{2ay}{e}$, the Rectangle of the Extremes will again be $=$ to the Rectangle of the Means, *i. e.* $\frac{2ayb}{e} = \frac{2aac+2acx-cee}{e}$; and, multiplying both Sides by e, $2ayb = 2aac + 2acx - cee$; which is the ſecond Equation.

4. *The Reduction of both Equations.* The firſt was $4ady = 2a^3 + 2axx + 2ayy$. Therefore dividing by $2a$, $2dy = aa + xx + yy$. And ſubſtracting $aa+yy$, $2dy - aa - yy = xx$.

The latter Equation was $2ayb = 2aac + 2acx - eec$, i. e. ſubſtituting again the Value ee, which was $aa+2ax+xx+yy$. $2ayb = 2aac + 2acx - aac - 2acx - cxx - cyy$, i. e. $2ayb = aac - cxx - cyy$; and by tranſpoſition, $cxx = aac - 2ayb - cyy$; and dividing by c, $xx = \frac{aac - 2ayb - cyy}{c}$, or $aa - yy - \frac{2ayb}{c}$, or (putting $2f$ for $\frac{2ab}{c}$) $xx = aa - yy - 2fy$.

Therefore we have the Value of xx twice expreſſed, but by Quantities partly unknown, becauſe y is found in both Equations. Whence now we muſt make a new Compariſon of their Values, in order to have this new,

5. *Third Equation*; in which there is only one of the unknown Quantities: $2dy - aa - yy = aa - yy - 2fy$; and adding on both Sides, both yy and aa, $2dy = 2aa - 2fy$; or dividing by 2, $dy = aa - fy$; and tranſpoſing fy, $dy + fy = aa$; and dividing by $d+f$, $y = \frac{aa}{d+f}$; which is the Value of the Quantity y in known Terms.

But this Value being ſubſtituted for y, and its Square for yy

in one of the precedent Equations, *viz.* in this, $xx=2dy-aa-yy$, will give $xx=\frac{2daa}{d+f}-aa-\frac{a^4}{dd+2df+ff}$; *i. e.* all being reduced to the ſame Denomination, $xx=\frac{aadd-aaff-a^4}{dd+2df+ff}$; and $x=\frac{\sqrt{aadd-aaff-a^4}}{dd+2df+ff}$.

6. *The Geometrical Conſtruction,* which *Schooten* gives us, *p.* 153. Having made the Angle KAB (*Fig.* 29. *N.* 2.) equal to the given one D, from A erect AL perpendicular to KA, meeting the Perpendicular EM in L; and from the Center L, with an Interval equal to the given right Line d, deſcribe a Circle, that ſhall cut KA and EL (produced) in K and M. Then aſſuming EN = KA, join MA, and from N draw NH parallel to it, which ſhall meet AB in H. Afterwards, from L, with the Interval LA, deſcribe the Segment of a Circle ACB, and from H draw HC perpendicular to AB, meeting the Circumference in C, and join AC, CB.

N. B. The reaſon of this elegant Conſtruction, which the Author concealed for the ſake of Learners, we will here ſhew: 1. Therefore he reduced the laſt Equation (extracting the Root, both of Numerator and Denominator, as well as it could bear) to this: $x=\frac{a}{d+f}$ multipl. by $\sqrt{dd-ff-aa}$; ſo that thereby the Conſtruction might be reduced to this Proportion, as $d+f$ to a, ſo is $\sqrt{dd-ff-aa}$ to x. 2. He made the Angle KAE = to the given one D, and the Angle KAL a right one, that ſo the Segment of a Circle being deſcribed from L, the Angle in that Segment might alſo be equal to the given one, according to 32 *Lib.* 3 *Eucl.* 3. By doing this, EL expreſſes the Quantity f, ſince the Triangles KOA (GFD, *N.* 1.) and AEL (for the Angles LAE and AKO are equal, becauſe each makes a right one with the ſame third Angle KAO) being ſimilar, you have

KO : OA :: AE : EL, *i. e.* f.

$$c \;:\; b \;::\; a \;:\; \frac{ab}{c}.$$

4. Now making LM and LK $=d$, you had EM $=d+f$, and AK $=\sqrt{dd-ff-aa}$ (for $\overline{AL}^2$ is $=aa+ff$, which being ſubſtracted from $\overline{LK}^2=dd$, there remains $\overline{AK}^2=dd-aa-ff$.)

5. Wherefore there now remained nothing to conſtruct the laſt Equation above, but to make EN $=$ AK, and to draw HN parallel to AM; for thus was the whole Proportion had, *viz.*

$$\mathrm{EM} : \mathrm{EA} :: \mathrm{EN} : \mathrm{EH}$$

$$d+f : a :: \sqrt{dd-ff-aa} : x. \quad \text{Q. E. F.}$$

For the Point H being determined, a Perpendicular HC thence erected in the Segment already deſcribed, determines the Point C, which anſwers the Queſtion.

PROBLEM VII.

THE four Sides of a Quadrangle to be inſcribed in a Circle, being given, to find the Diagonals and their Segments; and ſo to conſtruct the Quadrangle, and inſcribe it in the Circle. As, *e. g.* ſuppoſe the given Sides are AB, BC, CD, DA (*Fig.* 30. *N.* 1.) which now we ſuppoſe to be joined into a Quadrangle inſcrib'd in the Circle, and the Diagonals AC and BD being alſo drawn (*N.* 3.) to find firſt the Segments of the Diagonals A*e*, B*e*, *&c.* which being had, the Conſtruction is prepared.

SOLUTION.

Denomination. Make AB $= a$, BC $= b$, CD $= c$, DA $= d$, A*e* $= x$, [for this Segment alone being found, the reſt will be found alſo, as will be evident from the Proceſs.] Since therefore the Vertical Angles at *e* are equal, and likewiſe the Angles BCA, BDA; DCA, DBA in the ſame Segment, *&c.* are equal, the Triangles A*e*D and B*e*C; and A*e*B and C*e*D are ſimilar: wherefore it will follow that,

1. As DA to A*e* ſo CB to B*e*

$$d \ldots\ldots x \ldots\ b \ldots\ \frac{bx}{d}$$

2. As AB to B*e* ſo CD to C*e*

$$a \ldots\ \frac{bx}{d} \ldots\ c \ldots\ \frac{bcx}{ad}$$

3. As AB to A*e* ſo CD to D*e*

$$a \ldots\ldots x \ldots\ c \ldots\ \frac{cx}{a}$$

Therefore the whole Diagonal AC will be $= x + \frac{bcx}{ad}$, and BD $= \frac{bx}{d} + \frac{cx}{a}$.

2. *Equation.* But now by *Prop.* 48. *Lib.* 1. *Math. Enucl.* the Rectangle of the Diagonals is equal to the two Rectangles of the oppoſite Sides.

Diag. AC, $x + \frac{bcx}{ad}$

Diag. BD, $\frac{bx}{d} + \frac{cx}{a}$

$$\frac{cxx}{a} + \frac{bccxx}{aad}$$

$$\frac{bxx}{d} + \frac{bbcxx}{add}.$$

Therefore $\frac{bxx}{d} + \frac{cxx}{a} + \frac{bbcxx}{add} + \frac{bbcxx}{add}$

the ▭ of the Diagonals is $= ac + bd$.

3. *Reduction*, i. e. taking a for Unity,

$$\frac{bxx}{d} + cxx + \frac{bbcxx}{dd} + \frac{bccxx}{d} = c + bd;$$

i. e. the Quantities on the left hand being reduc'd to the ſame Denomination.

$$\frac{bdxx + cddxx + bbcxx + bccdxx}{dd} = c + bd;$$

and multiplying both Sides by dd,

$$bdxx + cdxx + bbcxx + bccdxx = cdd + bd^3;$$

and dividing both Sides by d

$$bxx + cdxx + \frac{bbcxx}{d} + bccxx = cd + bdd;$$

and then dividing both Sides by $b + cd + \frac{bbc}{d} + bcc$,

$$xx = \frac{cd + bdd}{b + cd + \frac{bbc}{d} + bcc};$$

i. e. in the preſent Caſe, where b by chance happens to be $= a$,

$$xx = \frac{cd + dd}{1 + cd + \frac{c}{d} + cc}$$

Therefore $x = \frac{\sqrt{cd + bdd}}{b + cd + \frac{bbc}{d} + bcc}.$

or in our Caſe $x = \dfrac{\sqrt{cd+dd}}{1+cd+\frac{c}{d}+cc.}$

4. *The Geometrical Conſtruction*, which, by ſuppoſing a (and in the preſent Caſe alſo b) to be Unity, ought to determine; 1. The Quantities cd, $\frac{c}{d}$, cc, and their Aggregate plus Unity. 2. The Aggregrate of cd and dd. 3. To divide the one by the other. And, 4. To extract the Root out of the Quotient, or alſo to extract the Roots firſt out of each Quantity, and divide them by one another; which may all of them be ſeparately done in ſo many ſeparate Diagrams, but more elegantly connected together after the following, or ſome ſuch like way. 1. Join AD and DC (*N.* 2.) into one Line, and having deſcribed a Semi-circle thereupon, erect the Perpendicular DE; and the Line AE drawn will $= \sqrt{cd+dd}$. 2. Taking the Angle CAG at pleaſure, make AF = AB, and joining DF, draw CG parallel to it; ſo that FG will be $= \frac{c}{d}$. 3. In the Vertical Angle make AH = CD, the Line HI drawn parallel to DF will cut off AI $= cd$. 4. If you take AL = AH or CD in the Perpendicular AK = AB, and join KH, and draw LM parallel to it, you'll have AM $= cc$. 5. Having prolong'd AG to N, and AH to O, ſo that GN be = AI + AM and AO = AB or AK, and having deſcrib'd a Semi-circle upon the whole Line NO, then a Perpendicular AP being erected, will be $= \sqrt{1+cd+\frac{c}{d}+cc}$; and ſo, 6. If AQ be made = AE and AR = AF or AB, and you join PQ and draw a line RS from R parallel to it; AS will be $= x$, *i. e.* the Segment ſought Ae of the Diagonal AC; which being given by virtue of the firſt Inference premis'd in the Denomination above, and drawing DS and (having made DT = BC) TV parallel to it; you'll have alſo the other Segment Be = SV, and by their Interſection on the Line AB (*N.* 3.) the Point e, thro' which the Diagonals muſt be drawn, which will be terminated by the other given Sides, and thence you'll have the quadrilateral Figure ABCD ſought, to be circumſcribed about the Circle, according to *Conſect.* 6. *Defin.* 8. *Matheſ. Enucl.*

NB. *But that we had here conſulted the Learner's Eaſe, the Artifice of this Conſtruction might have been propos'd after a more ſhort and occult way, thus:* Make DE a mean Proportional between AD and DC, and draw AE. Then having made any Angle CAG, make AF = AB, and with this Interval, deſcribe the Circle FROK, and join DF and draw CG parallel to it. Moreover, in the oppoſite Vertical Angle, having made AH = CD, draw HI parallel to DF, and having erected the Perpendicular AK, and thence cutting off AL = AH, make LM parallel to HK; again, having prolonged AH to O, and GN being taken equal to AI + AM, make AP a mean Proportional between AO and AN, cutting an occult Circle in R; and laſtly having made AQ = AE, if RS be drawn parallel to QP, you'll have AS the value of x ſought, *&c.*

III. *Some Examples of Affected Quadratick Equations.*

PROBLEM I.

Having given, to make a right-angled Triangle ABC, *the Differences between the leſſer and greater Side, and between the greater Side and the Hypothenuſe of a right-angled Triangle* ABC, *to find the Sides ſeparately, and form the Triangle.* E. g. Having given the right Line DB (*Fig.* 31.) for the Difference of the Perpendicular and Baſe, and CE for the Difference of the Baſe and Hypothenuſe, to find the Perpendicular AC; which being found, you'll have alſo, by what we have ſuppoſed, the Baſe AB, and the Hypothenuſe BC.

SOLUTION.

Make the Difference DB $= a$, CE $= b$, and put x for the Perpendicular; then the Baſe, which is greater than that, will be $x + a$, and the Hypothenuſe $x + a + b$. Therefore by virtue of the Pythagorick Theorem,

$$2xx + 2ax + aa = xx + 2ax + 2bx + aa + 2ab + bb;$$

and ſubſtracting from both Sides $xx + 2ax + aa$,

$$xx = 2bx + 2ab + bb.$$

Wherefore by the firſt Caſe of affected quadratick Equations

$$x = b + \sqrt{2ab + 2bb}.$$

Conſtruction. Find a Mean Proportional AK between AH $= 2b$ and AI $= a$ (*Fig.* 31. *N.* 2.) and having made both AF and AG $= b$, place the Hypothenuſe KF from A to L, and cut off GC equal to the Hypothenuſe GL; thus you'll have AC the Perpendicular of the Triangle ſought, and adding DB, you'll alſo have the Baſe AB, and from thence having drawn the Hypothenuſe BC, it will be found to differ from AB by the required Exceſs CE.

The Arithmetical Rule. Join twice the Product of the Differences multiplyed by one another, to twice the Square of the Difference of the Baſe and the Hypothenuſe; and if the ſquare Root of this Sum, being extracted, be added to the aforeſaid Difference, you'll have the Perpendicular ſought. Suppoſe, *e. g.* both the Differences of CE and DB = 10.

PROBLEM II.

THE Hypothenuſe and Sum of the Legs of a right-angled Triangle being given, to find the Sides. E. g. Let the Hypothenuſe BC be given (*Fig.* 32.) and the Sum of the Legs CAB, to find the Legs AB and AC ſeparately, and form the Triangle.

SOLUTION.

Make the Hypothenuſe BC $= a$, the Sum of the Legs $= b$. Make one Leg, *e.g.* AB $= x$; then will the other Leg AC be $= b - x$. Therefore

$2xx - 2bx + bb = aa$; and adding $2bx$, and taking away bb, $2xx = 2bx + aa - bb$; and dividing by 2,

$$xx = bx + \frac{aa - bb}{2}.$$

Therefore according to Caſe 1. of affected Quadraticks,

$$x = \tfrac{1}{2} b + \sqrt{\tfrac{1}{4} bb + \frac{aa - bb}{2}}$$

$$\text{i. e. } x = \tfrac{1}{2} b + \sqrt{\tfrac{1}{2} aa - \tfrac{1}{4} bb}$$
$$\text{or} = \tfrac{1}{2} b - \sqrt{\tfrac{1}{2} aa - \tfrac{1}{4} bb}.$$

The Geometrical Conſtruction. Having deſcribed a Semicircle upon BD $=$ BC $= a$, apply therein the equal Lines BE and DE, and having deſcribed another Semi-circle on BE, pply therein BF $= \tfrac{1}{2} b$, to be prolonged further out. Laſtly, another little Semi-circle be deſcribed with the Interval EF; the whole Line AB will be the affirmative Root or the Leg ſought, and GB the negative Root, &c.

The Arithmetical Rule. From half the Square of the Hypothenuſe, ſubſtract the fourth part of the Square of the iven Sum and the Root extracted out of the Remainder, if it be added to half the Sum, will give one Leg of the Triangle; and ſubſtracted from the given Sum, will give alſo the other. Suppoſe, *e. g.* BC to be 20, and the Sum of the Legs 28.

PROBLEM III.

THE Hypothenuſe, and the Difference BD *of the Legs of the ſame* △ *as above, being given* (*Fig.* 33.) *to find the Legs.*

SOLUTION.

Make the leſſer Leg $= x$, and the Difference of the Legs $= b$; then the greater Leg will be $x + b$. Let the Hypothenuſe be $= a$. Therefore

$2xx + 2bx + bb = aa$; and taking away $2bx + bb$,
$2xx = aa - 2bx = bb$; and dividing by 2,

$$xx = -bx + \frac{aa - bb}{2}.$$

Therefore by Caſe 2, $x = -\frac{1}{2}b + \sqrt{\frac{1}{4}bb + \frac{aa - bb}{2}}$.

i. e. $x = -\frac{1}{2}b + \sqrt{\frac{1}{2}aa - \frac{1}{4}bb}$.

The Geometrical Conſtruction. Having deſcrib'd a Semi-circle upon BD = BC or *a*, apply therein the equal Lines BC and DC, and having deſcrib'd another Semi-circle on DC, apply in it DF = $\frac{1}{2}b$; and if with this interval you cut off FA from FC, the remainder AC will be the leſſer Leg ſought, &c.

The Arithmetical Rule. From half the Square of the Hypothenuſe ſubſtract the Square of half the Difference, and i you take half the Difference from the Square Root of the Remainder, you'll have the leſſer Leg of the Triangle required, and by adding to it the given Difference, you'll have alſo the greater. *E. g.* Let the Hypothenuſe be 20, and the Difference of the Legs 4.

PROBLEM IV.

HAving given the Area of a right-angled Parallelogram and the Difference of the Sides; to find the Sides. E. g. I the Area be = to the Square of the given Line DF, and the Difference of the Sides be ED (*Fig.* 34.) to find the Sides o the Rectangle.

SO

SOLUTION.

Make the given Area $=aa$, the Difference of the Sides $=b$, and the leſſer Side x; then the greater will be $x+b$. Therefore the Area $xx+bx=aa$; and ſubſtracting bx,

$$xx=-bx+aa.$$

Therefore according to Caſe 2, $x=-\frac{1}{2}b+\sqrt{\frac{1}{4}bb+aa}$.

The Geometrical Conſtruction. Join at right Angles AG $=a$, and GH $=\frac{1}{2}b$, and having drawn AH and prolong'd it, deſcribe a little Circle with the interval GH: ſo you'll have AE the leſſer Side, and AD the greater of the Rectangle ſought, &c.

The Arithmetical Rule. Add the given Area and the Square of half the Difference into one Sum, then ſubſtract or add the Difference from or to the Root extracted; and ſo you'll have the greater and leſs Sides of the Rectangle.

PROBLEM V.

THE Difference between each of the Legs of a right-angled Triangle and the Hypothenuſe, being given, to find the Legs; and ſo the whole Triangle. E. g. Suppoſe the Difference of the leſs Leg to be BD, and of the greater DE (*N.* 1. *Fig.* 35.) to find the Legs themſelves, and ſo make the Triangle.

SOLUTION.

For BD put a, and for DE, b. Let the greater Side be x; the Hypothenuſe then will be $x+b$; therefore the leſſer Side will be $x+b-a$. Now the Squares of the Sides are = to the □ of the Hypothenuſe, *i. e.* $2xx-2ax+2bx+bb-2ab+aa=xx+2bx+bb$; and taking away $xx+2bx+bb$, $xx-2ax-2ab+aa=0$; and adding $2ax$ and $2ab$, and taking away aa,

$$xx=2ax+2ab-aa. \text{ Therefore}$$

$$x=a+\sqrt{aa+2ab-aa}; \text{ i. e.}$$

$$x=a+\sqrt{2ab}.$$

The Geometrical Conſtruction. Between the given Diffeences BD and DE, find (*N.* 2.) a mean Proportional DF, nd join to it at right Angles the equal Line FG, and cut ff DH equal to DG; and ſo you'll have BH the greater ide of the Triangle ſought. This being prolong'd to C, ſo at HC be $=b$, and having deſcribed a Semicircle upon the whole Line BC, apply therein BA = BH; and having drawn AC, the ſought Triangle ABC will be formed.

The Arithmetical Rule. If the ſquare Root, extracted from the double Rectangle of the Differences, be added to the greater Difference, you will have the greater Leg ſought, *&c.*

PROBLEM VI.

THE Sum of the Baſes of two unequal Rectangles of the ſame Height, being given, together with the Area of either, (viz. *the greater*) *and the Proportion of the Sides of the other* (viz. *the leaſt:*) *to find the Sides ſeparately.* *E. g.* Let the Sum of the Baſes be AB, (*Fig.* 36. *N.* 1.) and the Square of the Line BC = to the Area of the greater Rectangle; and let the Sides of the leſſer Rectangle be to one another as CD to DE: To find the Sides of both the Rectangles, *i. e.* to find the common Altitude; which being found, the other Sides will be eaſily obtain'd from the *Data*; or to find the Baſe of the greater, which, with the ſame eaſe, will diſcover the reſt.

SOLUTION.

Make AB $=a$, and the Area of the greater Rectangle $=bb$; and the Proportion of the Altitude to the Baſe in the leſſer, as c to d; to find, *e. g.* the greater Baſe, which call x. Then the common Altitude will be $=\frac{bb}{x}$, and the Baſe of the leſſer Rectangle $=a-x$. Wherefore you'll have fo the Equation, as c to d, ſo $\frac{bb}{x}$ to $a-x$. Therefore $ac-cx$ $=\frac{bbd}{x}$; and multipl. by x, $acx-cxx=bbd$; and adding cxx, and taking away bbd, $acx-bbd=cxx$. Now tha you may conveniently divide both Sides by c, make firſt a c to b, ſo is d to a fourth, which call f; and then put cf fo bd, and you'll have $acx-bcf=cxx$; and dividing by c $ax-bf=xx$; and ſo according to Caſe 3, $x=\frac{1}{2}a+$ $\sqrt{\frac{1}{4}aa-bf}$, or $\frac{1}{2}a-\sqrt{\frac{1}{4}aa-bf}$.

The Geometrical Conſtruction. Find firſt the Quantity (*Fig.* 36. *Num.* 2.) according to the following Proportion, a c to b, ſo d to f; and a mean Propoportional between b an f will be $=\sqrt{bf}$. Then having, with the Interval equal to $\frac{1}{2}$ de deſcribed a Semicircle (*N.* 3.) upon the given Line AB, and erected BD $= \sqrt{bf}$ perpendicular, and having made EF $=$ to it, CF will be $\sqrt{\frac{1}{4}aa - bf}$: To which AC being added, will give one Value of x, and FB the other. And for the common Altitude, which we called $\frac{bb}{x}$, make as x to b, ſo b to a fourth, *i. e.* as AF to FH, ſo FH to FG; which will be the Altitude of both Rectangles Ag and Bg, which may now eaſily be conſtructed.

The Arithmetical Rule might eaſily be had from this Equation reduced; but you may have it more commodiouſly from this other

SOLUTION.

Let the Denomination remain the ſame as above, only here put x for the common Altitude, and expreſs the Ratio of the leſſer Baſe of the Rectangle to this Altitude by e, and that Baſe will be $= ex$: Therefore the Baſe of the greater Rectangle will be $= a - ex$. Having now multiplied the common Altitude by the greater Baſe, the Area of the greater Rectangle will be $ax - exx$, and hence you'll have the Equation $ax - exx = bb$; and adding exx, and taking away bb, $ax - bb = exx$; and dividing by e, $\frac{ax - bb}{e} = xx$. Therefore by Caſe 3, $x = \frac{a}{2e} + \sqrt{\frac{aa}{4ee} - \frac{bb}{e}}$, or $x = \frac{a}{2e} - \sqrt{\frac{aa}{4ee} - \frac{bb}{e}}$.

Wherefore now this will be the *Arithmetical Rule*. If from the fourth Part of the Square of the Sum of the Baſes, divided by the □ of the Name of the Ratio, you ſubſtract the given Area divided by the Name of the Ratio, and if the Root extracted out of the Remainder be added to or ſubſtracted from half the Sum of the Baſes divided by the ſame Name of the Ratio, this Sum or Remainder will give the Altitude of the given Rectangles; which, multiplied by the Name of the Ratio, will give one of the Baſes: And that Baſe being ſubſtracted from the given Sum of the Baſes, will give the other Baſe. For example, let the Sum of the Baſes be 16, the Area of one of the Rectangles 30, the Name of the Ratio, which the common Altitude has to the Baſe of the other Rectangle, $=2$. Then will 5 or 3 b the common Altitude, *&c.*

PROBLEM VII.

THE Perpendicular of a right-angled Triangle let fal from the right Angle, and its Baſe being given: to fin the Segments of the Baſe, and ſo form the Triangle.

E.g. Let the Baſe of the right-angled Triangle you are to form be AB, (*Fig.* 37.) and BE or BF the Length of the Perpendicular; to find the Segments of the Baſe, and ſo the Point D, from which you are to draw the Perpendicular CD, to form the Triangle ABG.

SOLUTION.

Let the given Baſe be $=a$; and the given Perpendicular $=b$: And let one of the Segments of the Baſe be $=x$; then will the other $=a-x$, and b a mean Proportional between the ſaid Segments; *i. e.* x to b, as b to $a-x$: Therefore $ax-xx=bb$; and by adding xx, and taking away bb, $ax-bb=xx$. Therefore by Caſe 3, $x=\frac{1}{2}a+\sqrt{\frac{1}{4}aa-bb}$, or $\frac{1}{2}a-\sqrt{\frac{1}{4}aa-bb}$.

The Geometrical Conſtruction. Having deſcribed a Semicircle upon the given Line AB, if you erect the Perpendicular BE, and from the Point G (which determines EG parallel to AB) let fall GD equal to it, you will have the two Segments ſought; *viz.* $AD=\frac{1}{2}a+\sqrt{\frac{1}{4}aa-bb}$, and $DB=\frac{1}{2}a-\sqrt{\frac{1}{4}aa-bb}$, which Conſtruction is evident to any attentive Perſon, even without the Analyſis.

But that Caſe may, by the by, be taken notice of, wherein the given Perpendicular would not be BE, but BF. For in this Caſe, the Perpendicular BF being erected upon AB, the Parallel FG would not cut the Semicircle; which is an infallible Sign that the Problem in this Caſe is impoſſible, ſince the Perpendicular is ſuppoſed to be greater than half the Baſe; which is inconſiſtent with a right Angle.

The Arithmetical Rule. From the Square of half the Baſe, take the Square of the given Perpendicular, and add or ſubſtract the ſquare Root extracted out of the Remainder, to or from half the Baſe; and on the one hand, the Sum will give the greater Segment; and on the other, the Difference will give the leſs.

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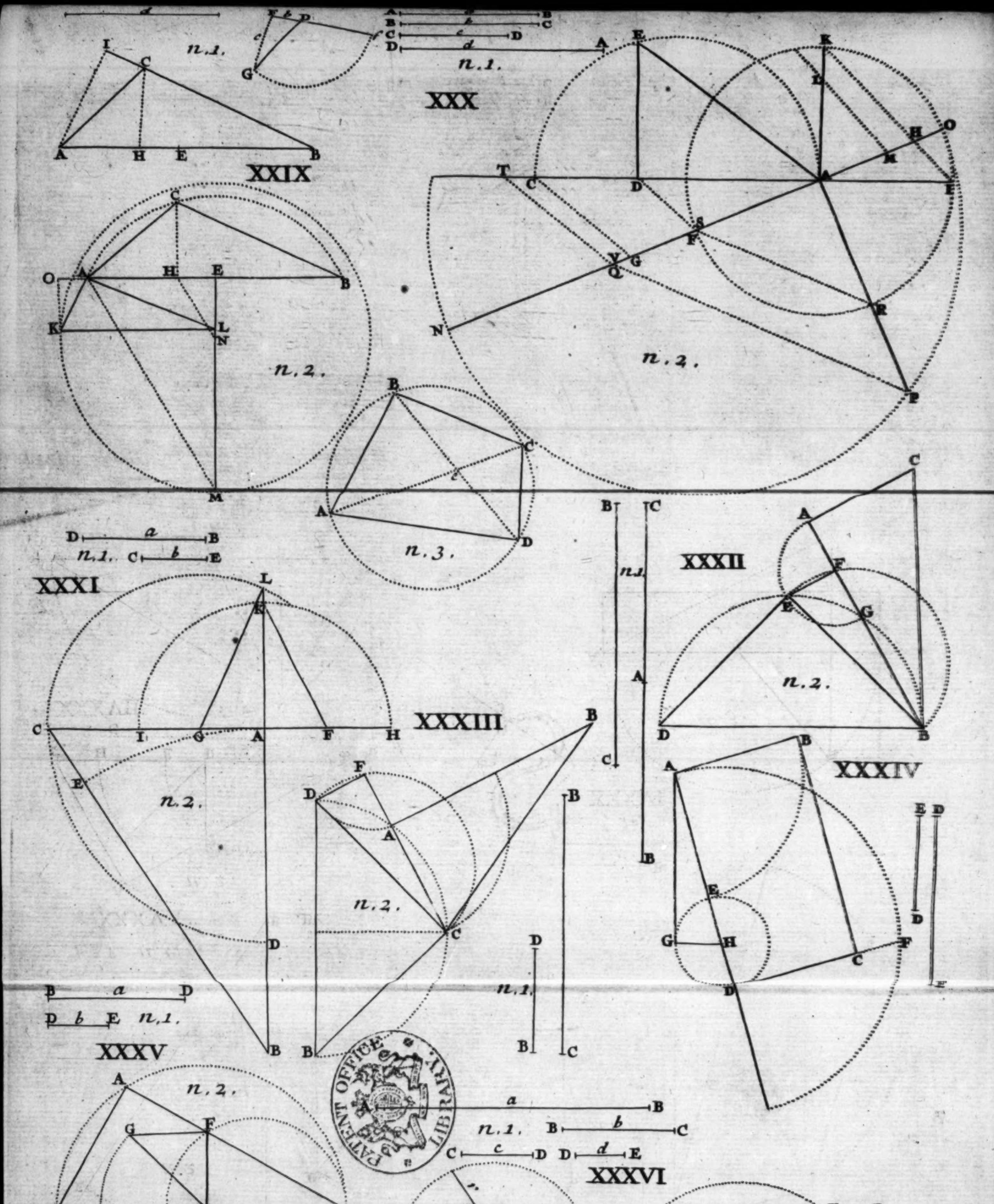

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PROBLEM VIII.

Having given the Perpendicular of a right-angled Triangle that is to be let fall from the right Angle, and the Difference of the Segments of the Baſe; to find the Segments, and deſcribe the Triangle.

E.g. If the Perpendicular is, as above, BE, and the Difference of the Segments AH (*Fig.* 38. *N.* 1.) to find the Segments AD and DB, from whoſe common Bound you re to erect a Perpendicular DG or DC to form the Triangle.

SOLUTION.

Make the leſſer Segment $=x$, and the Difference of the Segments $=a$, then the greater Segment will be $x+a$. Make the given Perpendicular, as before, $=b$: Therefore you'll have, as $x+a$ to b, ſo b to x; and conſequently, $xx+ax=bb$; and ſubſtracting ax, $xx=bb-ax$. Wherefore, according to Caſe 2, $x=-\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}$.

The Geometrical Conſtruction. Make HD $=\frac{1}{2}a$, DG $=b$, and perpendicular to HD; then HG will be $=\sqrt{\frac{1}{4}aa+bb}$ $=$ HB, or HA, *viz.* having drawn a Semicircle from H thro' G. Therefore DB is the leſs Segment, and AD the greater; and having drawn AG and BG, or (making the Perpendicular DC $=$ DG) and having drawn AC and BC on the contrary ſide AB, the Triangle will be conſtructed. Or, with *Des Cartes*, make (*N.* 2.) HE $=\frac{1}{2}a$, and EB $=b$; and having deſcribed a Circle from H thro' E, draw BHA; and ſo you'll have the two Segments ſought, AB being the greater, and DB the leſſer.

The Arithmetical Rule. Join the Squares of the half Difference and Perpendicular into one Sum, and then having extracted the Root, ſubſtract half the Difference from it, and the Remainder will be the leſſer Segment ſought; and having added the Difference, you will have alſo the greater.

PROBLEM IX.

Having given, for a right-angled Triangle, one Segment of the Baſe, and the Side adjacent to the other Segment; to find the reſt, and deſcribe the Triangle.

If the leſſer given Segment of the Baſe be DB, (*Fig.* 39. *N.* 1.) and the Side AC adjacent to the other Segment; to find the greater Segment of the Baſe; which being found, the reſt are eaſily obtain'd, and conſequently the whole Triangle.

SOLUTION.

Make the greater Segment $= b$, and the given Side $= c$, the Segment ſought $= x$. Now if we ſuppoſe the Triangle ABC to be already found, it is evident, 1. If from the Square of AC you ſubſtract the Square of AD, you'll have the □ of CD $= cc - xx$. 2. The ſame □ of CD may alſo be otherwiſe hence obtain'd; becauſe, the Angle at C being a right one, CD is a mean Proportional between BD and DA, *i. e.* between b and x; whence the Rectangle of the Extremes bx is $=$ □ of the Mean CD. Wherefore now it follows, 3. That $cc - xx = bx$; and adding xx, $cc = bx + xx$; and ſubſtracting bx, $-bx + cc = xx$. Therefore, according to Caſe 2, $x = -\frac{1}{2}b + \sqrt{\frac{1}{4}bb + cc}$.

Geometrical Conſtruction. Join EF $= \frac{1}{2}b$ (*Fig.* 39. *N.* 2.) and FA $= c$ at right Angles, and having deſcribed a Circle from E thro' F, draw AEB; ſo you'll have DA the greater Segment, and DB the leſs: having erected therefore a Perpendicular from D, and deſcribed a Semicircle upon AB, you'll have C the Vertex of the Triangle ſought; from whence you are to draw the Sides AC and BC.

The Arithmetical Rule. Join the □ of half the given Segment, and the □ of the given Side into one Sum; and having extracted the Root of it, if you thence take half the given Segment, you'll have the Segment ſought.

PROBLEM X.

THE perpendicular Height, and the Difference of the Segments of the Baſe, and the Difference of the other Sides of an oblique-angled Triangle, being given: to find the Sides, and form the Triangle.

As, if the Altitude CD be given, (*Fig.* 40. *N.* 1.) and alſo the Difference of the Segments of the Baſe EB, and the Difference of the Sides FB (as is evident from the Triangle ABC (*N.* 2.) conceived to be formed beforehand:) and you are to determine the Baſe itſelf, and both Sides, &c.

SOLUTION.

Make the given Perpendicular CD $= a$, (See *Fig.* 40. *N.* 2.) EB $= b$, FB $= c$: For the leſſer Segment of the Baſe AD put x, and the greater will be $x + b$. It is now evident, that you may obtain the □ of CB by the Addition of the Squares of DC and of BD, *viz.* $aa + xx + 2bx + bb$;

and the □ of AC by the Addition of the Squares of AD and DC, *viz.* $aa + xx$: So that the Side AC will be $= \sqrt{aa+xx}$, and the Side BC $= \sqrt{aa + xx + 2bx + bb}$. But ſince alſo this ſame Side BC may be obtained, by adding the Difference c to the Side AC, ſo that it be $= c + \sqrt{aa+xx}$: you'll have this Equation, $c + \sqrt{aa + xx} = \sqrt{aa + xx + 2bx + bb}$; and ſquaring both Sides, $cc + aa + xx + \sqrt{4ccaa + 4ccxx} = aa + xx + 2bx + bb$; and ſubſtracting from both Sides $cc + aa + xx$, $\sqrt{4ccaa + 4ccxx} = 2bx + bb - cc$; and again ſquaring it, $4ccaa + 4ccxx = 4bbxx + 4b^3x - 4bccx + b^4 - 2bbcc + c^4$; and ſubſtracting from both Sides $4ccxx$, (becauſe c is leſs than b) and tranſpoſing the reſt,

$$\begin{array}{l} 4ccaa - 4b^3x - b^4 \\ \qquad + 4bccx + 2bbcc = 4bbxx - 4ccxx; \\ \qquad\qquad - c^4 \end{array}$$

and dividing by $4bb - 4cc$,

$$\frac{\begin{array}{l} 4cccaa - 4b^3x - b^4 \\ \qquad + 4bccx + 2bbcc = xx. \\ \qquad\qquad - c^4 \end{array}}{4bb - cc,}$$

i. e. dividing the affected Quantities by 4, both above and underneath,

$$\begin{array}{l} -b^3x + \dfrac{4ccaa - b^4 + 2bbcc - c^4}{4bb - 4cc} = xx \\ \dfrac{+bccx}{bb - cc}, \end{array}$$

and actually dividing the former Part by $bb - cc$ $-bx + \dfrac{4ccaa - b^4 + 2bbcc - c^4}{4bb - 4cc} = xx$. Therefore, according to the ſecond Caſe, $-\frac{1}{2}b + \sqrt{\frac{1}{4}bb + \dfrac{4ccaa - b^4 + 2bbcc - c^4}{4bb - 4cc}}$ $= x$; or reducing $\frac{1}{4}bb$ to the ſame Denomination with the reſt,

$$-\tfrac{1}{2}b + \sqrt{b^4 \frac{-bbcc + 4ccaa - b^4 + 2bbcc - c^4}{4bb - cc}} = x;$$

and leaving out thoſe Quantities that deſtroy one another,

$$-\tfrac{1}{2}b + \sqrt{\frac{4ccaa + bbcc - c^4}{4bb - 4cc}} = x.$$

The Arithmetical Rule. Multiply four times the Square of FB by the Square of the Perpendicular CD, and add to it the Product of the Square of EB into the Square of FB, and from the Sum ſubſtract the Biquadrate of FB; and note the Remainder. Then ſubſtract four times the Square of FB from four times the Square of EB; and alſo note the Remainder. Laſtly, divide the firſt noted Remainder by the ſecond, and from the ſquare Root of the Quotient take half EB: Thus you'll have the leſſer Segment of the Baſe ſought, *&c.* *E. g.* In Numbers you may put 2 for FB, 4 for EB, and 12 for CD.

As for the *Geometrical Conſtruction*, the Quantity of the laſt Equation contain'd under the radical Sine will help us to this Proportion, $4bb - 4cc : 4aa + bb - cc :: cc$: a fourth; or dividing all by 4, $bb - cc : aa + \frac{1}{4}bb - \frac{1}{4}cc :: \frac{1}{4}cc$: a fourth, which is $\frac{1}{4}$ of the Quantity under the radical Sine. Aſſuming therefore the Quantity c for Unity, make (*N.* 3.) $IK = c$, IN and $KL = b$; then NO will be $= bb$, and ſubſtracting $OP = cc$, (*i. e.* Unity) there will remain $NP = bb - cc$. In like manner, make IS and $KM = a$, then ST will $= aa$; to which if you add $SX = \frac{1}{4}NO$, and take thence $XV = \frac{1}{4}$ Unity; TV will be $= aa + \frac{1}{4}bb - \frac{1}{4}cc$. Wherefore if you make NR equal to TV, and $PQ = \frac{1}{4}$ of Unity or cc, ſince NP is $= bb - cc$; by the Rule of Proportion there will come out $DR = \frac{1}{4}$ of that Quantity, which is under the radical Sine. Therefore this being taken four times, will give DZ for the whole Quantity; to which if you join DY = to Unity, and, having deſcribed a Semicircle upon the whole Line YZ, erect the Perpendicular DE; this will be the Root of the ſaid Quantity, and taking hence moreover $EF = \frac{1}{2}b$, you'll have DE or DA the leſſer Segment of the Baſe ſought. Therefore adding $GB = b$ to DG, DB will be the greater Segment; and, having let fall the Perpendicular $DC = a$, BC and AC will be the Sides ſought. Q. E. F.

IV. *Some Examples of Affected Biquadratick Equations, being like Affected Quadratick ones.*

PROBLEM I.

TO find a Square ABCD (*ſuch as in the mean while we'll ſuppoſe to be in* Fig. 41. N. 1.) *from which having taken away another Square* AEFG, *which ſhall be half the former, there will be left the Rectangle* GC *of a given Area.* *E. g.* Suppoſe the given Area equal to the Square of the given Line LM, to find the true Sides of the Squares AB and AE, anſwering to theſe ſuppoſed ones, *N.* 1.

SOLUTION.

Make the Area of the Rectangle that is to remain$=bb$, and $GB=x$; then BC or AB will be $=\frac{bb}{x}$, and ſubſtracting GB from it, the remaining Side of the leſſer Square will be $AG=\frac{bb}{x}-x$, i. e. $\frac{bb-xx}{x}$. Since therefore the Square of this is ſuppoſed to be half of the Square of AB, this will be the Equation: $\frac{b^4-2bbxx+x^4}{xx}=\frac{b^4}{2xx}$; and multiplying by xx, $b^4-2bbxx+x^4=\frac{b}{2}$; and multiplying by 2, $2b^4-4bbxx+2x^4=b^4$; and ſubſtracting $2b^4$, and adding $4bbxx$, $2x^4=4bbxx-b^4$; and dividing by 2, $x^4=2bbxx-\frac{b^4}{2}$.

N. B. *The ſame Equation may be obtain'd, if, putting* x *for* GB *or* FH, *and having found the* □ *of* AG *or* GF, *as above, you make* $\frac{b^4-2bbxx+x^4}{xx}=2bb-xx$.

This laſt Equation, tho' it be a Biquadratick, yet may be rightly eſteemed only a quadratick one, becauſe there is neither x^3 nor ſingle x in it; and ſo you may ſubſtitute this for it, $yy=2bby-\frac{b^4}{2}$, *viz.* by ſuppoſing $y=xx$. Whence, according to the third Caſe of affected Quadraticks, y will $=bb+\sqrt{b^4-\frac{b^4}{2}}$, i. e. $\sqrt{\frac{b^4}{2}}$, or $=bb-\sqrt{\frac{b^4}{2}}$. Therefore $x=\sqrt{bb+\sqrt{\frac{b^4}{2}}}$.

Geometrical Conſtruction. Now if the given Line b be aſſumed for Unity, bb and b^4 will be $=$ to the ſame Line. Therefore, if between LM as Unity, and $MN=\frac{1}{2}b$, viz. $\frac{b^4}{2}$, you find a mean Proportional MO (*Fig.* 41. *N.* 2.) that will be $=\sqrt{\frac{b^4}{2}}$, which being ſubſtracted from LM, and added to it, will give the two Values of the Quantity y. Therefore by extracting its Roots, *i. e.* by finding other mean Proportionals LR and LS between the Quantities found LP and LQ, and Unity, (*N.* 3.) they will be the two Values of the Quantity x ſought; the firſt whereof LR will ſatisfy the Queſtion, and the other LS will imply an Impoſſibility. Wherefore to form the □ itſelf, ſince its Side will be $=\frac{bb}{x}$; by making (*N.* 4.) $x : b :: b :$ a fourth, it will be obtained. And this may be further prov'd, if you find a mean Proportional BK between BI = LR and the Side of the □ BC, and it be equal to the given Quantity LM.

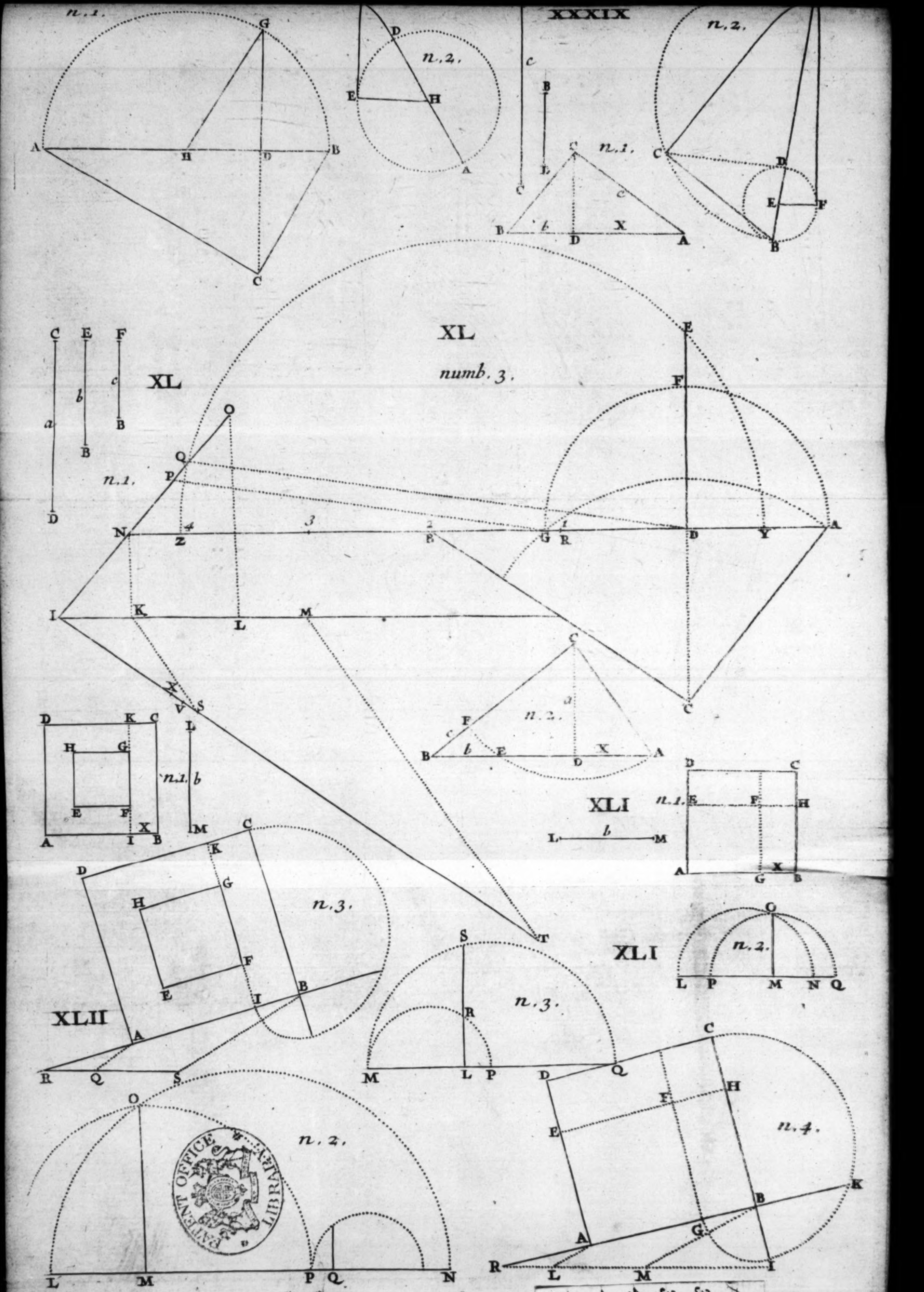
Arithmetical Rule. From the given Area or the Square of the given Line LM, ſubſtract the Root of half the Biquadrate of the ſame Line; thus you will have the Value of the □ FC, *viz.* xx: Therefore extracting further the ſquare Root of this, it will be the Value of x ſought.

PROBLEM II.

TO find another Square ABCD (Fig. 42. N. 1.) *out of the middle, whereof if you take another Square* EFGH, *which ſhall be a fourth Part of the former; the Area of the Rectangle* BK *contained under* BC *and* FG *prolonged, ſhall be equal to the Square of a given Line* LM; *i. e.* having theſe given, to find the Segment BI, and conſequently alſo the Side BC or AB.

SOLUTION.

Make the Area of the given Rectangle, or the Square of LM = to bb, and the ſought Side of the Rectangle BI $= x$; then the other Side BC will be $=\frac{bb}{x}$, and having ſubſtracted out of it IF and GK (*i. e.* $2x$) the Side of the leſſer Square FG will $=\frac{bb}{x}-2x$, i. e. $\frac{bb-2xx}{x}$; and becauſe the Square of this is to be equal to the fourth part of the greater □, by the Hypotheſis, therefore $\frac{4b^4-16bbxx+4x^4}{xx}=\frac{b^4}{xx}$; and multi-



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multiplying both Sides by xx, $4b^4 - 16bbxx + 4x^4 = b^4$; and taking away $4b^4$, and adding $16bbxx$, $4x^4 = 16bbxx - 3b^4$; and dividing by 4, $x^4 = 4bbxx - \frac{3}{4}b^4$. Therefore according to the third Caſe of affected quadratick Equations, $xx = 2bb \pm \sqrt{4b^4 - \frac{3}{4}b^4}$ i. e. $xx = 2bb \pm \sqrt{3^{1} b^4}$. Therefore $x = \sqrt{2bb \pm \sqrt{3\frac{1}{4}b^4}}$.

Geometrical Conſtruction. If the given Line b be taken for Unity, b^4 and bb will be equal to it. Therefore if between LM as Unity, and MN $= 3\frac{1}{4}b$, you find a mean Proportional MO (*N.* 2. *Fig.* 42.); 'twill be $\sqrt{3\frac{1}{4}b^4}$; which ſubſtracted from MQ $= 2b$, or added to it, will give two Values of the Quantity xx, *viz.* PQ and IQ; the firſt whereof will be an affirmative one, and of uſe here. Therefore a mean Proportional QR found between PQ and Unity, will expreſs the Quantity ſought x.

Therefore for forming the Square itſelf, ſince its Side AB is $= \frac{bb}{x}$, you may proceed as in the former Conſtruction, (*vid. N.* 3.)

PROBLEM III.

Having given the Baſe of a right-angled Triangle, and a mean Proportional between the Hypothenuſe and Perpendicular, to find the Triangle. As if the given Baſe be AB (*Fig.* 43.) and the mean Proportional between AC and BC be CD; to find the Perpendicular BC, and Hypothenuſe AC.

SOLUTION.

Make the given Baſe $= a$, and the Mean proportional $= b$, the Perpendicular BC $= x$, then will the Hypothenuſe be, by the Hypoth. $\frac{bb}{x}$. Therefore $\frac{b^4}{xx} = aa + xx$; and multiplying both Sides by xx, $b^4 = aaxx + x^4$; and ſubſtracting $aaxx$, $b^4 - aaxx = x^4$. Therefore by the ſecond Caſe, of affected Quadraticks $xx = -\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + b^4}$, and $x = \sqrt{-\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + b^4}}$.

Or thus:

Make the Hypothenuſe AC $= x$, then will the Perpendicular BC be $= \frac{bb}{x}$. Therefore $xx = aa + \frac{b^4}{xx}$; and multiplying by xx, $x^4 = aaxx + b^4$. Therefore by Caſe 1. $xx = \frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + b^4}$ and $x = \sqrt{\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + b^4}}$.

Geometrical Conſtruction; firſt, for the latter Equation. If a be put for Unity, the Line AB will be alſo $= aa$, and making, $a : b :: b :$ a third, *i. e.* LM : MN :: LO : OP, and you'll have bb. Having erected the Perpendicular AQ = OP upon AM, and drawn MQ or Mn equal to it; $\sqrt{\frac{1}{4}a^4 + b^4}$, and conſequently A$n$ will be $= \frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + b^4}$, *i. e.* the value of xx. Moreover a Mean proportional AC found between An and AR Unity, will be the value of x, *i. e.* the Hypothenuſe ſought, which being found, you may eaſily compleat the Triangle ABC.

2. For the former Equation, making every thing as before, AK would be the value of the Quantity $xx +$ *i. e.* $-\frac{1}{2}a$, $+ \sqrt{\frac{1}{4}a^4 + b^4}$. Therefore a mean Proportional TR found between RS = AK and AR Unity, will be the value of x, *i. e.* the Perpendicular ſought and ſo AT will be the Hypothenuſe of the Triangle ſought.

Arithmetical Rule. In the firſt Solution, add the Biquadrate of the given mean Proportional to the Biquadrate of half the given Baſe; and having extracted the ſquare Root of the Sum, take from it half the Square of the given Baſe; the Root of the Remainder will give the Perpendicular of the Triangle ſought. In the ſecond, add half of the Square of the given Baſe to the aforeſaid Root, and the Root of the Sum will give the Hypoth. ſought.

PROBLEM IV.

Having the Hypothenuſe of a right-angled Triangle given, and a mean Proportional between the Sides; to find the Triangle. As if the Hypothenuſe be AC (*Fig.* 44.) and a mean Proportional between the Sides BD, to find the Sides AB and BC.

SOLUTION.

Make the given Hypothenuſe $= a$, and the mean Proportional $= b$, and the Perpendicular BC $= x$; then the Baſe

AB by the Hypoth. will be $\frac{bb}{x}$. Therefore $\frac{b}{xx} + xx = aa$ and multiplying by xx, $b^4 + x^4 = aaxx$; and ſubſtracting b^4, $x^4 = aaxx - b^4$. Therefore by the third Caſe, $xx = \frac{1}{2}aa \pm \sqrt{\frac{1}{4}a^4 - b^4}$, and $x = \sqrt{\frac{1}{2}aa \pm \sqrt{\frac{1}{4}a^4 - b^4}}$.

Geometrical Conſtruction. If a be put for Unity, AC will be alſo $= aa$, and by making as AC to CG (a to b) ſo AF to GH (b to a third) this third will be GH $= bb$. Aſſuming therefore OC $= \frac{1}{2}aa =$ OB the Radius of a Semicircle, and having erected CD $= bb =$ BE parallel to it; EO will be $\sqrt{\frac{1}{4}a^4 - b^4}$, and conſequently EC $= \frac{1}{2}a - \sqrt{\frac{1}{2}a^4 - b^4}$, and EA $= \frac{1}{2}a + \sqrt{\frac{1}{4}a^4 - b^4}$, *viz.* the two Values of the Quantity xx. Therefore to get the two Values of x, you muſt extract the Roots of them, *i. e.* you muſt find the mean Proportionals AL and AM between Unity AC and AI = EC on the one Side, and AK = AE on the other; altho' theſe laſt may be more compendiouſly had, and the Triangle itſelf immediately conſtructed, if having found EC and EA, you draw CB and AB: For theſe will be thoſe two laſt mean Proportionals = AL and AM; for becauſe of the Triangles ABC, AEB, and BEC, BC is a mean Proportional between AC and CE, and AB a mean Proportional between the ſame AC and AE, by *Prop.* 8. *Lib.* 6. *Eucl.* which is *Conſect.* 3. *Schol.* 2. *Prop.* 34. *Lib.* 1. *Math. Enucl.*

PROBLEM V.

H*Aving given the Area and Diagonal of a right-angled Parallelogram, to find the Sides, and ſo the Parallelogram.* As if the given Area be = to the Square of a given Line BD (*Fig.* 45.) and the Diagonal be AC; to find the Sides AB and BC.

SOLUTION.

If for the given Area, or Square of the Line BD you put bb, and make the Diagonal AC $= a$, and for the leſſer Side BC, put x; the other Side will be $\frac{bb}{x}$. Therefore $\frac{b^4}{xx} + xx = aa$; and multiplying by xx, $b^4 + x^4 = aaxx$; and ſubſtracting b^4, $x^4 = aax - b^4$. Which Equation, ſince it is the ſame with that of the preceding Problem (which is no wonder, ſince this Problem perfectly coincides with the fourth; for the Diagonal AC is the Hypothenuſe, and BD, whoſe

Square is = to the given Area of the Rectangle, is a mean Proportional between the Sides AB and BC) and ſo will have the ſame Conſtruction, (ſee *Fig.* 45.) and the ſame Arithmetical Rule, which may be eaſily formed from the laſt Equation of the preceding Problem.

PROBLEM VI.

THE firſt of three proportional Lines being given, and another Line whoſe Square is equal to both the Squares of the other two, to find thoſe two Proportionals. As if AC (*Fig.* 46.) be the firſt of the three Proportionals, and ED be another Line given, whoſe Square is equal to the two Squares of the others taken together; to find thoſe two as being ſecond and third Proportionals.

SOLUTION.

If for AC you put a, and make the given Line ED $= c$, and the ſecond Proportional $= x$; the third will be $\frac{xx}{a}$. Wherefore the Squares of the two laſt will be $\frac{x^4}{aa} + xx = cc$, $=$ ☐ of ED, by the Hypoth. and multiplying both Sides by aa, $x^4 + aaxx = aacc$; and ſubſtracting $aaxx$, $x^4 = -aaxx + aacc$. Therefore $xx = -\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + aacc}$, and $x = \sqrt{-\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + aacc}}$.

Geometrical Conſtruction. If a be put for Unity, then AC will alſo $= aa$ and a^4, and making as AC to CD (a to c) ſo AF to DE (c to a third) DE will be $= cc$. Now having made AK = DE, *i. e.* cc; a mean Proportional AI found between AC and AK will be $\sqrt{aacc}$. Therefore taking AO $= \frac{1}{2}$AC, *viz.* $\frac{1}{2}aa$, the Hypothenuſe OI will be $= \sqrt{\frac{1}{4}a^4 + aacc}$. And OA $\frac{1}{2}a$ being ſubſtracted from OI or OH equal to it, will leave AH the value of xx; and the Root of that being extracted, *i. e.* finding another mean Proportional AG between AC and AH, and it will be the value of x, *i. e.* the ſecond of the Proportionals ſought; and ſince AC is the firſt given one, AH will be the third. Q. E. F.

NB. This Conſtruction may be abbreviated, and the firſt Operation, by which you find DE, to which afterwards AK is made equal, may be omitted. For ſince you make uſe of a mean Proportional between CA and DE ſought, which is =

to the given Line ED, and afterwards AI a mean Proportional between AC and AK is ſought; it is evident that AI will be equal to ED which is given, and conſequently that they may be immediately joined at right Angles at the begining of the given Line AC, and the reſt may then be done as before.

Some Examples of Cubick and Biquadratick Equations, both ſimple and affected, whether reducible or not.

PROBLEM. I.

Between two given right Lines to find two mean Proportionals. E. g. Suppoſe given AB the firſt, and CD the fourth, (*Fig.* 47. *N.* 1.): between theſe to find two mean Proportionals.

SOLUTION.

Make the firſt of the given Quantities AB $= a$, the other CD $= q$, and the firſt Mean $= x$; then will the latter be $\frac{xx}{a}$, and conſequenly $\frac{x^3}{aa} = q$; and multiplying by aa, $x^3 = ** aaq$. The Central Rule will be $\frac{L}{2} =$ AD, $\frac{r}{2L}^2 =$ DH. i. e. according to a Suppoſition we ſhall by and by make, $\frac{1}{2}a$ $\frac{1}{2}a =$ AD, and $\frac{1}{2}q. =$ DH.

Geometrical Conſtruction. If AB or a be made Unity, and alſo the *Latus Rectum* of your Parabola, and by means of this *Latus Rectum*, you deſcribe the Parabola, according to *Schol.* 1. *Prop.* 1. *Lib.* 2. *Math. Enucl.* [ſee *N.* 2. and 3. *Fig.* 47.] in which AB is the *Latus Rectum*; A1, A2, &c. the Abſciſſæ; AI, AII, &c. the Semi-ordinates; and if you make (*N.* 4.) AD $= \frac{1}{2}a$, and from D having erected a Perpendicular $= \frac{1}{2}q$, deſcribe a Circle at the interval AH, cutting the Parabola in N: Which being done, a Perpendicular NO to the Axis will be the Root ſought, or the Value of x, *i. e.* the firſt of the Means, and conſequently OA the other; ſince NO by the firſt Property of the Parabola (ſee *Prop.* 1. *Lib.* 2. *Matheſ. Enucl*) is a mean Proportional between the *Latus Rectum* AB, and the Abſciſſa AO. And by this means there will come out, by *Baker*'s Central Rule, the very Conſtruction of *Des Cartes*, *Geom.* p. m. 91.

The Arithmetical Rule. Multiply the Square of the firſt by the fourth given; and the Cube Root extracted out of the Product, will expreſs the firſt of the Means ſought.

PROBLEM II.

THE ſolid Content of a Parallelepipedon, and the Proportion of the Sides, being given, to find the Sides. As, if the given Capacity or ſolid Content be = to the Cube of a certain given Line IK (*Fig.* 48. *N.* 1.) and the Proportion of the Height to the Length be as AB to BC, and to the Breadth as the ſame AB to BD; to find firſt the Altitude, which being had, the other Dimenſions will alſo be known, by the given Proportions.

SOLUTION.

Make IK $= a$, AB $= b$, BC $= c$, and BD $= d$; and laſtly the Height ſought $= x$, then as b to c, ſo x to the Length required $\frac{cx}{b}$; and as b to d, ſo x to the Breadth ſought $\frac{dx}{b}$. Multiplying therefore theſe three Dimenſions of the Parallelepipedon together, and you'll have its Capacity or ſolid Content $\frac{cdx^3}{bb} = a^3$; and multiplying by bb, $cdx^3 = a^3bb$; and dividing by cd, $x^3 = \frac{a^3bb}{cd}$ i. e. $x^3 ** - \frac{a^3bb}{cd} = 0$. Therefore the Central Rule will be the ſame as above, $\frac{L}{2} =$ AD and $\frac{r}{2L}{}^2 =$ DH, *i. e.* according to the Suppoſition which will by and by follow, $\frac{a}{2} =$ AD and $\frac{a^3bb}{2cd} =$ DH

Geometrical Conſtruction. If IK or a be made Unity, and at the ſame time the *Latus Rectum*, and by means of it you deſcribe a Parabola, after the way we have ſhewn, *Fig.* 47. *N.* 2. and 3. and ſhall always hereafter make uſe of; and then to prepare the Quantity $\frac{a^3bb}{2cd}$ (which in the Central Rule is the Quantity $\frac{r}{2}$) make (*N.* 2.)

IK : IM = BC :: KL = BD : MN

as *a* to *c* . . . . . . ſo *d* . . . . . to *e*; ſo that for *cd* you may put *ae*, and afterwards divide by *a* both above and underneath; you'll have the Quantity $\frac{r}{2} = \frac{aabb}{2e}$. Therefore by further inferring as 2*e* to *bb*, ſo *aa* to a fourth,

IO : IP = IT :: IK : IQ, and you will have the Quantity DH, which will determine the Center, after AD is made = to $\frac{a}{2}$. Therefore from that Center having deſcribed a Circle through the Vertex of the Parabola A (*N.* 3.) a Semi-ordinate NO drawn from the Interſection will be the Altitude ſought, which will eaſily give you the Length and Breadth by the Ratio's above given.

Another Solution,

Which will be more accommodated to the Arithmetical Rule.

Let the reſt of our Poſitions or *Data* remain as above, but the Name of the Proportion, which the Altitude has to the Length, let be $= e$, and of that which it has to the Breadth $= i$; then the Length will be $= ex$, and the Breadth $= ix$. Wherefore multiplying the Sides together, you'll have the whole Solidity $eix^3 = a^3$; and dividing by *ei*, $x^3 = \frac{a^3}{ei}$. Therefore $x = \sqrt{C.\frac{a^3}{ei}}$. Hence

The Arithmetical Rule. Multiply the given Names of the Ratio's together, and divide the given Cube by the Product; which done, the cubick Root extracted out of the Quotient will be the Altitude of the Solid ſought.

Another *Geometrical Conſtruction.* Now if we would alſo conſtruct this Equation $x^3 = \frac{a^3}{ei}$ geometrically, putting AB $= b$ for Unity; then BC and BD will be the Names of the Ratio's $= e$ and i. Making therefore firſt

IK : IM :: KL : MN

as *a* to *e*, ſo is *i* to a fourth *f*. (*Fig.* 49. *N.* 1.) Then *af* will be $= ei$, and the propoſed Equation will have this form: $x^3 = \frac{a^3}{af}$ i. e. $\frac{aa}{f}$. Therefore (2.) making as *f* to *a*, ſo *a* to a third IQ; this will be the Value of x^3. But (3.) by extracting the Cube Root of it, *i. e.* by finding two mean Proportionals between Unity b, *viz.* AB, and the Line found IQ; the firſt of them will give the Root ſought.

N. B. The ſame Conſtruction would come out according to *Baker*'s central Rule, which would have the ſame form of the Equation, as in the precedent Example $x^3 ** - \frac{aa}{f} = o$, i. e.

$$\text{taking } b \text{ for the Lat. Rect. and Unity} \begin{cases} \frac{L}{2} = AD, \text{ and } \frac{r}{2L^2} = DH \\ \frac{b}{2} = AD, \text{ and } \frac{aa}{2f}, \text{ i. e. } \tfrac{1}{2}IQ = DH. \end{cases}$$

(See *Fig.* 49. *N.* 2.)

PROBLEM III.

Having given the ſolid Content of a Parallelepipedon, and the Difference of the Sides; to find the Sides. As, if the given Capacity be equal to the Cube of any given Line LM, (*Fig.* 50. *N.* 1.) and the Difference whereby the Length exceeds the Breadth $=$ NO, and the Difference by which the Breadth exceeds the Altitude or Depth $=$ PQ; to find the Length, Breadth, and Depth.

SOLUTION.

Make the Side of the given Cube $= a$, the Exceſs of the Length above the Breadth NO $= b$, and of the Breadth above the Depth PQ $= c$, make the Depth $= x$; then will the Breadth $= x + c$, and the Length $= x + b + c$. Multiplying therefore theſe three Dimenſions together.

$$\begin{array}{ll}
\text{Length} & x + b + c \\
\text{Breadth} & x + c \\ \hline
 & xx + bx + cx \\
 & \quad\; cx + cb + cc \\ \hline
 & xx + 2cx + bx + cb + cc \\
\text{Depth} & x \\ \hline
 & x^3 \begin{matrix} +2c \\ +b \end{matrix} \Big\} xx \begin{matrix} +cb \\ +cc \end{matrix} \Big\} x = a^3:
\end{array}$$

Or, according to the Forms of *Baker* and *Des Cartes*,

$$x^3 \begin{matrix} +b \\ +2c \end{matrix}\, xx \begin{matrix} +bc \\ +cc \end{matrix}\, x - a^3 = o.$$

Where-

fig.	pag
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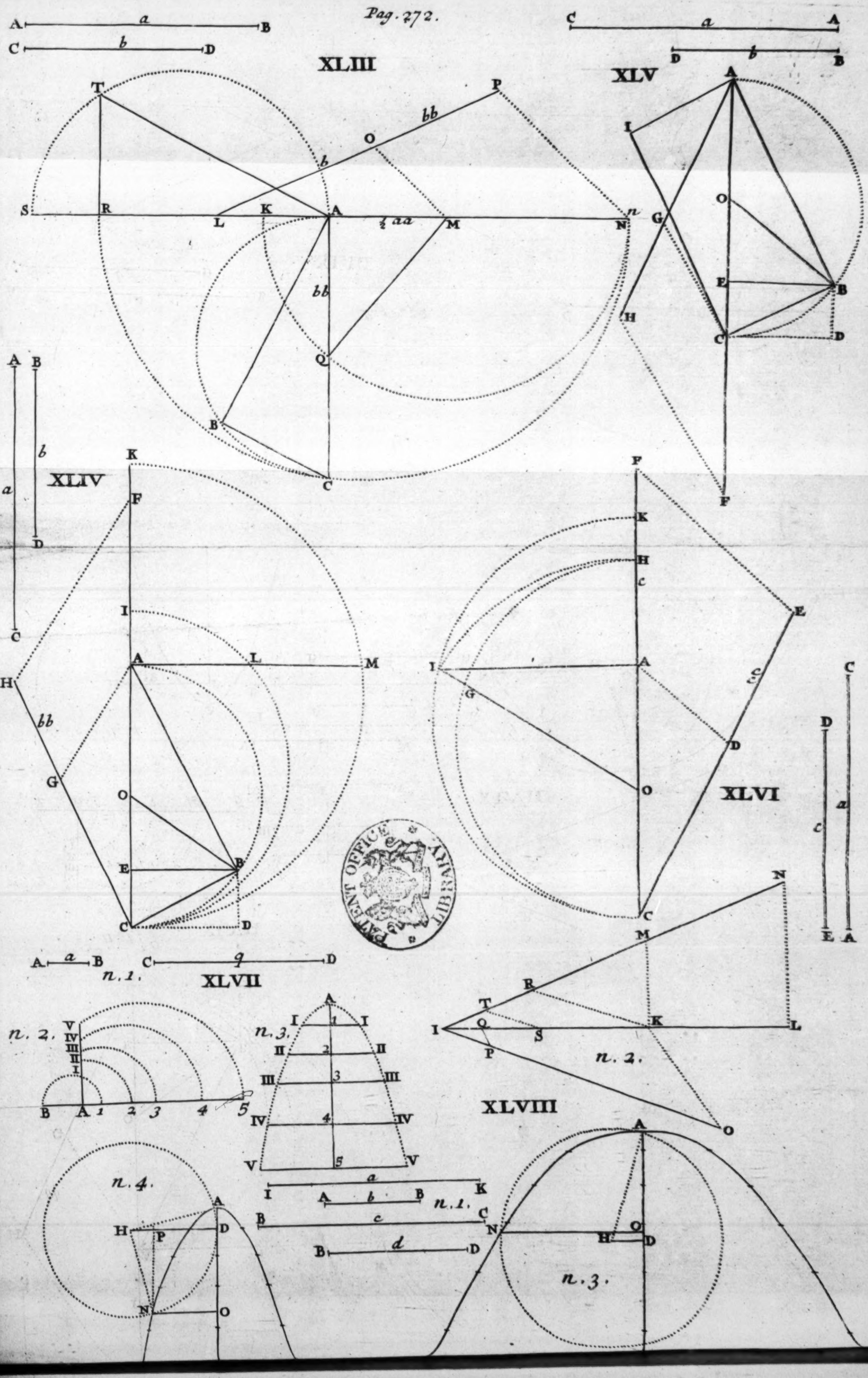

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i. e. by virt
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$\frac{a}{2} + \frac{bb +}{}$

$\frac{b^3 + 6bbc -}{2}$

Or more ſh

$\frac{bb}{8} =$ AD

$\frac{bbc - 3bc}{4}$

$\frac{bbc}{8} \backsim \frac{a^3}{2} =$

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Equation.

1. For A

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$\frac{1}{2}b + c$) ſo
will be $= \frac{p}{}$

BH to GI (
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Wherefore the central Rule contracted by the Suppoſition which will hereafter follow, will be this;

$$\frac{L}{2} + \frac{p^2}{8} \backsim \frac{q}{2} = AD,$$

and $\frac{p}{4} + \frac{p^3}{16} \backsim \frac{pq}{4} \backsim \frac{r}{2} = DH$;

i. e. by virtue of the Suppoſition juſt now mentioned (which takes LM, *viz.* *a* for Unity, and alſo for the *Lat. Rectum*) $\frac{a}{2} + \frac{bb + 4bc + 4cc}{8} \backsim \frac{bc \backsim cc}{2} = AD$. And $\frac{b + 2c}{4} + \frac{l^3 + 6bbc + 12bcc + 8c^3}{16} \backsim \frac{bbc \backsim 3bcc \backsim 2c^3}{4} \backsim \frac{a^3}{2} = DH$.

Or more ſhort; $\frac{a}{2} + \frac{bb}{8} + \frac{bc + cc}{2} - \frac{bc - cc}{2}$, i. e. $\frac{a}{2} + \frac{bb}{8} = AD$; and $\frac{b + 2c}{4} + \frac{b^3}{16} + \frac{1\frac{1}{2}bbc + 3bcc + 2c^3}{4} - \frac{bbc - 3bcc - 2c^3}{4} \backsim \frac{a^3}{2} = DH$, that is, $\frac{b + 2c}{4} + \frac{b^3}{16} + \frac{bbc}{8} \backsim \frac{a^3}{2} = DH$.

Geometrical Conſtruction. If LM or *a* be taken for Unity, and the *Latus Rectum* of the Parabola to be deſcribed; this being deſcribed, (*Fig.* 50. *N.* 3.) you are firſt of all to determine two Quantities AD and DH; which may be done two ways: either by *Baker*'s Form of his central Rule, or by ours immediately divided by the Quantities of the laſt Equation.

1. For AD, by our Form, $\frac{a}{2} + \frac{bb}{8} = AD$, you muſt make (*Fig.* 50. *N.* 1.) as *a* to *b*, ſo *b* to a third (AB to AC, ſo BE to CF) which will be *bb*, and D2 the eighth Part of this CF muſt be added to A2 the half of AB. And by *Baker*'s Form you muſt make, 1. As AB ($= a$, *N.* 2.) to AC ($= \frac{1}{2}p$, i. e. $\frac{1}{2}b + c$) ſo BE ($= \frac{1}{4}p$, i. e. $\frac{1}{4}b + \frac{1}{2}c$) to a fourth CF (which will be $= \frac{p^2}{8}$.) 2. Make moreover as AB to AG (*a* to *b*) ſo BH to GI (*c* to *bc*) and, as AB to AK (*a* to *c*) ſo BH to KL (*c* to *cc*) and the two Quantities found GI and KL (*bc* and *cc*) being added into one Sum, will give the Quantity $q =$

MN, the half whereof MO will expreſs the Quantity $\frac{q}{2}$ in the Rule, to be ſubſtracted from the former $\frac{a}{2}+\frac{p^2}{8}$. Therefore actually to determine the Quantity AD not in the Axis, but on another Diameter of the deſcribed Parabola, *N.* 5. (becauſe the Quantity p is in the Equation) having made a Perpendicular to the Axis aE $=\frac{p}{4}$, *i. e.* to the Line BE, *N.* 2. and from E having drawn EA parallel to the Axis, according to our Form AD, *N.* 1. muſt be laid off in the Diameter of the Parabola, *N.* 3. from A to D, either by Parts $\frac{a}{2}$, i. e. $\frac{1}{2}$LM from A to *c*, and $\frac{bb}{8}$, i. e. $\frac{1}{8}$CF, *N.* 1. from *c* to D: But according to *Baker*'s Form, firſt, you muſt put $\frac{a}{2}=\frac{1}{2}$LM, *N.* 1. from A to 1. Secondly, you muſt put from 1 to 2 the Quantity $\frac{p^2}{8}=$ CF, *N.* 2. Thirdly, from 2 to 3 backwards, you muſt put the Quantity to be ſubſtracted $\frac{q}{2}=$ MO, *N.* 2. which being done, the Point D will be determined.

[*It is evident, by comparing theſe two Ways of Conſtruction, that we may join our Forms not incommodiouſly to* Baker's; *becauſe, by ours, the Quantity* AD *was obtained more compendiouſly than by* Baker's, *which will alſo often happen hereafter. And where this Compendium cannot be had, there is another not inconſiderable one, that, if the Quantities* AD *and* DH, *determined according to both Ways, ſhall coincide, (which happens in the preſent Caſe) we may be ſo much the more ſure of the Truth.*]

2. As for DH, by our Form, you muſt put it from D to *e* in a Perpendicular erected from D to the left Hand, the Quantity $\frac{b+2c}{4}=$ BE, *N.* 2. falling here in the Axis. Then for the Quantity $\frac{b^2}{16}$, make (*N.* 4.) $a : \frac{1}{2}bb$ :: (LM to LN)

$\frac{1}{8}b$:

$\frac{1}{2}b : \frac{b^3}{16}$ (MO to NP) and this Quantity muſt be put from e to f in the Perpendicular to the Diam. Thirdly, for the Quantity $\frac{bbc}{8}$, you muſt farther make (*N.* 4.) $a : \frac{1}{2}bb$ :: (LM to LN) $\frac{1}{4}c$: a fourth (MQ to NR) which muſt be put from f to g. Laſtly, the Quantity $\frac{c^3}{2}$ (which is = MO, *N.* 2.) muſt be put backwards (becauſe to be ſubſtracted) from g to H, which is the Center ſought. In like manner, by *Baker*'s Form, firſt $\frac{p}{4}$ = BE is put from D to 1 even to the Axis. Secondly, for the Quantity $\frac{p^3}{16}$ make (*N.* 4.) $a : \frac{p^2}{8}$ :: (LM to LS = CF, *N.* 2.) $\frac{p}{2}$: a fourth (MT = AC *N.* 2. to SV) and this SV is further put (*N.* 3.) from 1 to 3. Thirdly, in the ſame *Fig. N.* 4. for the Quantity $\frac{pq}{4}$, make $a : \frac{p}{2}$ :: (LM to MT) $\frac{q}{2}$: to a fourth (LX to XZ) and this XZ is put backwards (*N.* 3.) from 2 to 3, which preciſely coincides with the Point g. Laſtly, the remaining Quantity $\frac{r}{2}$ (= MO, *N.* 2. and ſo by what we have ſaid above, preciſely coinciding with the Interval gH) is put backwards from g to H the Center ſought.

[*Hence it appears again, that* Baker's *Form is more laborious than ours; tho' both accurately agree, and hereafter, for the moſt part, we ſhall uſe them both together, tho' in the Work itſelf, rather in Figures, than in that tedious Proceſs of Words, which we have here for once made uſe of, that it might be as an Example for the following Conſtructions.*]

Having therefore found, by one or both Ways, the Center H, and thence deſcribed a Circle thro' the Vertex of the Parabola A; the Interſection N will give the Perpendicular to the Diameter NO, the Value of the Quantity x ſought.

PROBLEM IV.

To divide a given Angle NOP, *or a given Arch* NQTP (Fig. 51. N. 1.) *into three equal Parts*; *i. e.* the Radius NO being given, or aſſumed at pleaſure, and conſequently alſo the Chord of the Arch NP; to find NQ the Chord of the third Part of the given Arch.

SOLUTION.

If NO be made Unity, NP $= q$, and NQ be ſuppoſed $= z$; then QS being drawn parallel to TO, you'll have three ſimilar Triangles NOQ, QNR, and RQS. For ſince the Angle QOP is double of the Angle QON, and the ſame (as being at the Center) double alſo of the Angle at the Periphery QNR; therefore QNR will be equal to the Angle QON. But the Angle at Q is common to both Triangles: Therefore the Triangles NOQ, QNR, are equi-angular, and conſequently the Legs NQ and NR equal, as well as NO and QO; and by the like reaſon, alſo PY and PT. Wherefore, if RS be added to RY, the Line NP, by this Addition, will be triple of the Line NQ; and ſo will give an Equation, if RS be determined; which may be done by means of the △ QRS, ſimilar to the two former NOQ and QNR; for the Angle RQS is equal to the alternate one QOT = QNR, and the Angle at R common to the Triangles QNR and RQS, *&c.* Wherefore

NO : NQ :: NQ : QR

$$1 : z :: z : zz,$$

and NQ : QR :: QR : RS

$$z : zz :: zz : \frac{z^4}{z} = z^3.$$

Therefore, according to what we have above ſaid

$$q + z^3 = 3z; \text{ and ſubſtracting } q$$

$$z^3 = 3z - q; \text{ or}$$

$$z^3 - 3z + q = 0.$$

Therefore the central Rule will be (ſuppoſing NO for the *Latus Rectum* to be alſo Unity)

$$\frac{L}{2} + \frac{q}{2} = AD, \textit{ i.e.} \text{ by our Form, } \frac{1}{2} + \frac{3}{2}, \textit{ i.e. } 2NO = AD$$

and $\frac{r}{2} = DH.$ and $\frac{q}{2} = DH.$

Geometrical Conſtruction. Having deſcribed your Parabola, (*Fig.* 51. *N.* 2.) take in its Axis (becauſe the Quantity p is wanting in the Equation) AD = 2NO, and from D having erected a Perpendicular = $\frac{1}{2}$NP to H; that will be the Center, from which a Circle deſcribed thro' A, by cutting the Parabola in three Points, will give the three Roots of the Equation; *viz.* NO and *no* affirmative ones: the firſt whereof will expreſs the Quantity ſought NQ (*N.* 1.) The latter, the Line NV, being the Chord of the third Part of the Compl. of the Arch; and MO will expreſs the negative Root, which is equal to the former two taken together: All the ſame, as in *Des Cartes*, p. 91. but here ſomewhat plainer and eaſier.

PROBLEM V.

THREE Sides of a quadrilateral Figure to be inſcribed in a Circle, viz. AB, BC, CD, (Fig. 52. N. 1.) *being given; to find the fourth Side, which ſhall be the Diameter of the Circle.*

SOLUTION.

If we conſider the Buſineſs as already done, and make AB $= a$, BC $= b$, CD $= c$, and AD $= y$; we ſhall have firſt in the right-angled △ ABD, $\overline{BD}^2 = yy - aa$, and (ſince in the obtuſe-angled △ BCD, $\overline{BD}^2$ is $= \overline{BC}^2 + \overline{CD}^2 +$ 2BC × CE) if $\overline{BC}^2 + \overline{CD}^2$ (*i. e.* $bb + cc$) be ſubſtracted from $\overline{BD}^2$ ($yy - aa$) you'll have, (2.) $yy - aa - bb - cc =$ 2BC × CE. But 2BC × CE may alſo be otherwiſe obtain'd, (3.) If the Segment CE be otherwiſe determined; which may be done by help of the ſimilar Triangles ABD and CED, (for the Angles at B and E are right ones, and ECD and BAD equal, becauſe each with the ſame third BCD makes two right ones; the one ECD, by reaſon of its Contiguity; the other at A, by *lib.* 3. *Prop.* 22. *Eucl.*) *viz.* by ſaying,

as DA to AB ſo DC to CE

$y : a :: c : \frac{ac}{y}$; for now multiplying CE $= \frac{ac}{y}$ by BC $= b$, and then BC × CE will be $= \frac{abc}{y}$, and two ſuch $= \frac{2abc}{y}$. Now therefore $yy - aa - bb - cc = \frac{2abc}{y}$;

and

and multipl. by y,

$$\left.\begin{array}{l} y^3 * -aa \\ \quad -bb \\ \quad -cc \end{array}\right\} y = 2abc;$$ *i. e.* by *Baker*'s and *Des Cartes*'s Forms.

$$\left.\begin{array}{l} y^3 * -aa \\ \quad -bb \\ \quad -cc \end{array}\right\} y - 2abc = o.$$

Therefore the Central Rule will be (ſuppoſing the ſame Quantity, *e. g.* a, for Unity and the *Lat. Rect.*)

$\frac{L}{2} + \frac{q}{2} =$ AD, and

$\frac{r}{2} =$ DH, *i. e.* according to our Form,

$\frac{a}{2} + \frac{aa+bb}{2}\frac{+cc}{}$, i. e. $\frac{a+bb+cc}{2} =$ AD, and $\frac{2abc}{2}$, i.e. $bc =$ DH.

Geometrical Conſtruction, which, without any Circumlocution, from our Form is founded on the foll. in *Fig.* 52. *N.* 2.

LM $= a$, *N.* 3. A 1 $=$ LM from *N.* 2.
MN and LP $= b$ 1, 2 $= \frac{1}{2}$PQ
PQ $= bb$, 2, 3 $= \frac{1}{2}$ST
LS and MO $= c$, DH $=$ PR
PR $= bc$ MO and *mo* the two negative Roots
ST $= cc$ NO the affirmative Root; upon which,

having deſcribed a Semicircle, the quadrilateral Figure will be eaſily made. According to *Baker*'s Form, for AD you muſt firſt have made (*N.* 3.) A$c = \frac{1}{2}$LM, then cD $= \frac{q}{2} =$ VX, half the Line VZ, which is compounded of LM, PQ, and ST; but DH $=$ PR as above.

PROBLEM VI.

THE leaſt Side BA (Fig. 53. N. 1.) *and the Difference of the Segments of the Baſe of a right-angled Triangle being given; to find the Difference of the Sides, and ſo form the Triangle.* If we repreſent the Buſineſs as already done, AB and EC being given; to find FC.

SOLUTION.

Make AB $= a$, and EC $= b$, and FC $= x$; then will BC $a + x$: Therefore the $\overline{AC}^2 = 2aa + 2ax + xx$, and the

Line AC $= \sqrt{2aa + 2ax + xx}$, and AE $= \sqrt{2aa + 2ax + xx} - b$. Now therefore ACE, *i. e.* $\sqrt{2aa + 2ax + xx}$ multiply'd by b or $\sqrt{bb}$, i. e. $\sqrt{2aabb + 2abbx + bbxx}$ is $=$ GCF [but GC is $= 2a + x$] i. e. $2ax + xx$ by *Conſ.* 1. *Prop.* 47. *lib.* 1. *Matheſ. Enucl.* and ſquaring both Sides

$$2aabb + 2abbx + bbxx = 4aaxx + 4ax^3 + x^4;$$

and tranſpoſing all,

$$\begin{array}{lllll} x^4 + 4ax^3 & + 4aaxx & - 2abbx & - 2aabb & = 0. \\ & - bb & & & \\ (p) & (q) & (r) & (S) & \end{array}$$

Therefore (taking a for 1, and the *Latus Rectum* of the Parabola) the central Rule will be,

$$\frac{L}{2} + \frac{p^2}{8} - \frac{q}{2} = AD$$

and $\frac{p}{4} + \frac{p^3}{16} - \frac{pq}{4} - \frac{r}{2} = DH$, *i. e.* according to our Form and Reduction,

$$\frac{a}{2} + \frac{16aa}{8} - \frac{4aa + bb}{2}, \text{ i.e. } \frac{a}{2} + \frac{bb}{2} = AD, \text{ and } a = DH.$$

Geometrical Conſtruction. Firſt, from our Form, the Compendiouſneſs whereof will here appear; for it requires only one Preparation in which LM $= a$, (*N.* 2.) MN and LO $= b$, OP $= bb$, which being premiſed, and the Diameter AY (becauſe the Quantity p is in the Equation) being drawn (*N.* 3.) make A1 $= \frac{1}{2}$LM, and 1, 2, or 1D $= \frac{1}{2}$OP, and DH $=$ LM. The reſt therefore being alſo perform'd, which the Quantity S occurring in the preſent Equation requires, according to the laſt Precepts of our Introduction, you'll have NO the Value of x ſought; whence (*N.* 4.) with the Interval AB having deſcribed a Circle, and made a right Angle at B, if FC be made $=$ to NO, you'll have the $\triangle$ ABC required; and EC will be found to be of the Magnitude preſcribed at firſt.

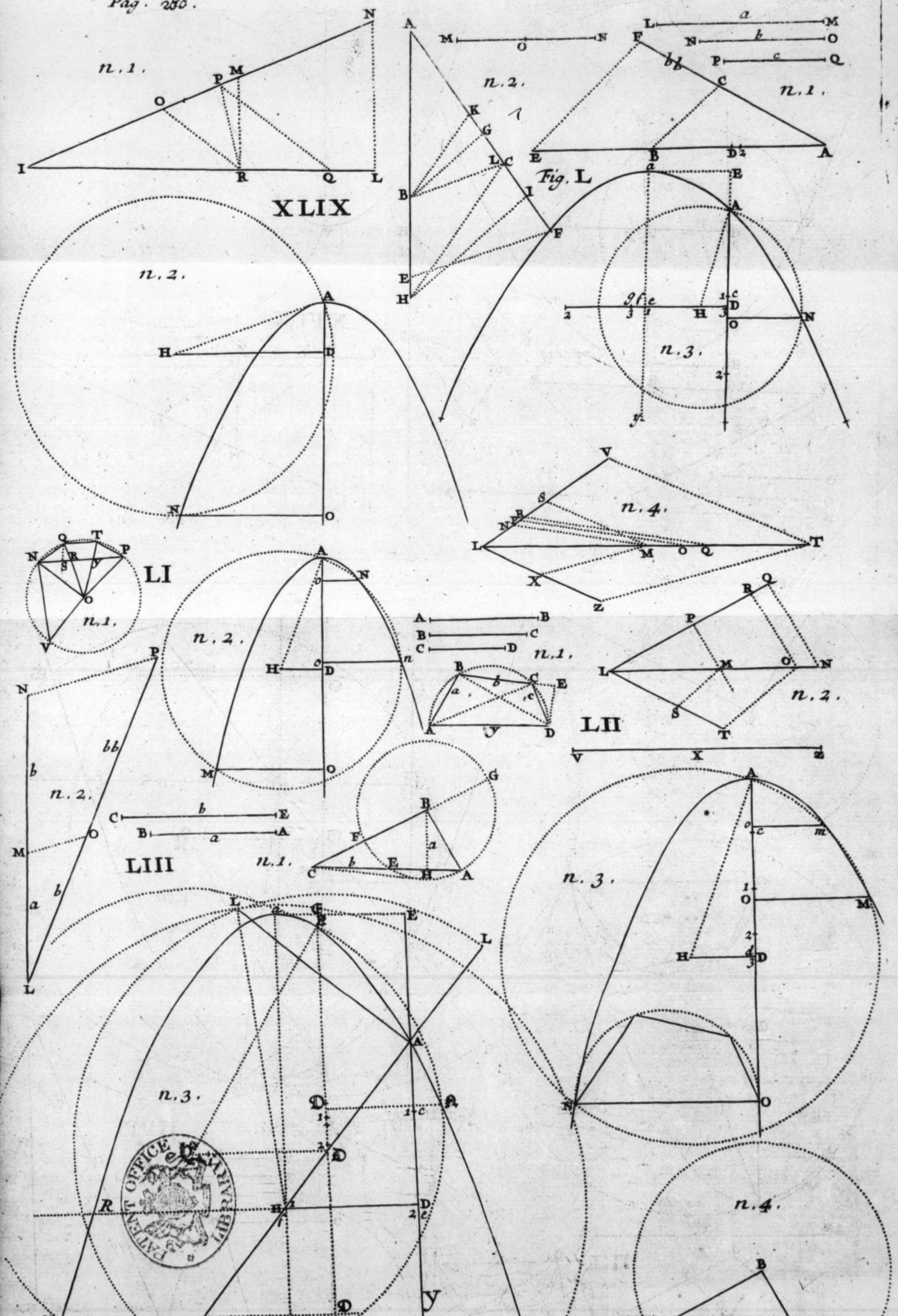
Now if you were to find the Center H by *Baker*'s Form without our Reduction, 1. You muſt lay off (*N.* 3.) $\frac{1}{2}$AB from A to *c*. 2. For the Quantity $\frac{p^2}{8}$ make $1 : \frac{p}{2} :: \frac{p}{4} :$ a fourth, which would be $=$ 2AB, *viz.* 2LM to be laid off from *c* to *d*. 3. The Quantity $\frac{q}{2}$ (to obtain which, you muſt ſubſtract OP (*N.* 2.) from the Quadruple of LM, and divide the Remainder by 2) muſt be ſet off backwards from *d* to *e*, for thus determining the Point D. 4. The Quantity $\frac{p}{4}$, which here is preciſely $=$ LM, muſt be transferr'd from D to *f* in the Perpendicular erected from D. 5. For the Quantity $\frac{p^2}{16}$ make $1 : \frac{p}{8} :: (= \frac{2AB}{2LM})$ $\frac{p}{2}$ (alſo $= \frac{2AB}{2LM}$) : a fourth, Quadruple of LM; and this muſt further be produced from *f* to the Point *g* (which here the Paper will not permit.) 6. For the Quantity $\frac{pq}{4}$ make $1 : \frac{p}{2} :: \frac{q}{2}$: a fourth (which would be $=$ to the Quadruple of LM, but OP being taken away) which muſt be ſet backwards from *g* to *h*. 7. Laſtly the Quantity $\frac{r}{2}$ ($=$ OP) ſet off from *h* backwards, or towards the right Hand to *i*, will at length give the Point H required.

Another Solution of the ſame Problem.

This Problem may be more eaſily ſolved, and will give a far more ſimple Equation, if you are to find not FC, but AE. Make therefore AE (*Fig.* 53. *N.* 1.) $= x$, and the reſt as above; then AC will be $= x + b$, and $\overline{AC}^2 = xx + 2bx + bb$; therefore $\overline{BC}^2 = xx + 2bx + bb - aa$; therefore the Square of the Tangent $HC = xx + 2bx + bb - 2aa$. But the Rectangle ACE will be $bb + bx$. Therefore by *Prop.* 47. *lib.* 1. *Math. Enucl.* $xx + 2bx + bb - 2aa = bb + bx$; and turning all over to the left Hand, $xx + bx - 2aa = 0$; or $xx = -bx + 2aa$. Therefore, by Caſe 2. of affected Quadraticks, $x = -\frac{1}{2}b + \sqrt{\frac{1}{4}bb + 2aa}$.

The Geometrical Conſtruction may be performed according to the Rules of Quadraticks, (*Fig.* 54. *N.* 1.) as will be evident to any attentive Reader. Having therefore deſcribed a Circle at the Interval BA, whether it be done from any arbitrary Center, (ſee *Fig.* 53. *N.* 4.) or upon AE found in the preſent *Fig.* making an Interſection at the ſaid Interval in B; and applying AE, and producing it until EC becomes equal to the given Quantity *q*, and at length having drawn BC, you'll have the Triangle right-angled at B, and alſo the Difference of the Sides FC. But to make it more ſhort and elegant; having determined AE by a little Circle,

fig.	pag.
49	271
50	272
51	276
52	277
53	278



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(*Fig.* 54. *N.* 1.)
cumference in C,
of the little Circ

Now if you w
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tral Rule will be

$\frac{L}{2}+\frac{p^2}{8}+\frac{q}{2}$

and $\frac{p}{4}+\frac{p^2}{16}+$

$\frac{a}{2}+\frac{bb}{8}+aa,$

and $\frac{b}{4}+\frac{b^2}{16}+$

The *Conſtruct*

N. 2.) $=a$, M

MN$=$PR, *viz.*

$a\mathfrak{E}=\frac{1}{4}b$, $\mathfrak{A}1=$
MO, 1, 2, or $1\mathfrak{P}$
thro' $\mathfrak{A}$, you'll
ſought; and $\mathfrak{MP}$

N. B. *Hence i*
veral Solutions a
others more com
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of Learners.

A *Right Line*
is is require
of BA *ſhall be to*

Since the firſt
firſt *a*, and the ot

(*Fig.* 54. *N.* 1.) prolong it to the oppoſite Part of the Circumference in C, and draw AB and CB; for as the Radius of the little Circle is $\frac{1}{2}b$, ſo EC is $= b$.

Now if you would conſtruct the Equation above by *Baker's* Rule (that its Univerſality may alſo be confirmed by an Example in Quadraticks) *viz.* $x^2 + bx - 2aa = 0$: The central Rule will be (taking a for 1 and the *L. R.*)

$$\frac{L}{2} + \frac{p^2}{8} + \frac{q}{2} = AD,$$

and $\frac{p}{4} + \frac{p^2}{16} + \frac{pq}{4} = DH$, *i. e.* by our Reduction,

$$\frac{a}{2} + \frac{bb}{8} + aa, \text{ i. e. } 1\tfrac{1}{2}a + \frac{bb}{8} = AD,$$

and $\frac{b}{4} + \frac{b^2}{16} + \frac{2aab}{4}$, i.e. $\frac{3b}{4} + \frac{b^2}{16} = DH$.

The *Conſtruction* therefore will be thus: LM (*Fig.* 54. *N.* 2.) $= a$, MO $= \frac{1}{4}b$, LP $= \frac{1}{2}b$, therefore PR $= \frac{bb}{8}$: MN $=$ PR, *viz.* $\frac{bb}{8}$, therefore PQ $= \frac{b^2}{16}$. In *Fig.* 53. *N.* 3. $a\mathfrak{E} = \frac{1}{4}b$, $\mathfrak{A}1 = \frac{1}{2}$LM, 1, 2, or $1\mathfrak{D} =$ PR; $\mathfrak{D}1 =$ LP $+$ MO, 1, 2, or $1\mathfrak{N} =$ PQ. Having drawn a Circle from $\mathfrak{N}$ thro' $\mathfrak{A}$, you'll have the affirmative Root $\mathfrak{R}\mathfrak{D} =$ to AE ſought; and $\mathfrak{M}\mathfrak{D}$ the negative Root.

N. B. *Hence it is evident, that one Problem may have ſeveral Solutions and Conſtructions, ſome more eaſy and ſimple, others more compound and laborious*; viz. *according as the unknown Quantity is aſſumed more or leſs commodious to the purpoſe: which may not be amiſs here to note for the ſake of Learners.*

PROBLEM VII.

A *Right Line* BD (Fig. 55.) *being any how divided in* A, *is is required to divide it again in* C, *ſo that the Square of* BA *ſhall be to the Square of* AC, *as* AC *to* CD.

SOLUTION.

Since the firſt Segments BA and AD are given, call the firſt a, and the other b, and call the ſought Quantity AC, x;

then CD will be $=b-x$. Now therefore ſince we ſuppoſe

$\overline{AB}^2 : \overline{AC}^2 :: AC : CD$

$aa : xx :: x : b-x$

$aab-aax$ will $=x^3$, and transferring the Quantities on the left Hand to the right, $x^3 * aax - aab = 0$.

Wherefore the central Rule (taking a for 1 and the $L.R.$) will be $\frac{1}{2}-\frac{q}{2}=$ AD, and $\frac{r}{2}=$DH, *i. e.* by our Form, $\frac{a}{2}-\frac{aa}{2}$, i. e. $0=$ AD, that D may fall on the Vertex of the Parabola; and $\frac{aab}{2}$, i. e. $\frac{b}{2}=$ DH. The Conſtruction therefore will be very ſimple, as is evident from the fifty-fifth Figure.

PROBLEM VIII.

There is given AB *the capital Line of a Horn-work*(Fig. 56. N. 1.) *and the Gorge* AD, *alſo part of the Line of Defence* EF, *to find the Face* BE, *the Flank* DE, *the Curtain (or the Chord)* DF, *alſo the Angle of the Baſtion* ABE, *&c. and ſo the whole Delineation of the Horn-work.* It is evident, if you have the Flank DE, or the Curtain DF, the reſt will be had alſo. Suppoſe therefore, the capital Line AB, and the Gorge AD, and part of the Line of Defence EF to be of the Magnitudes denoted by the Letters a, b, c, on the right Hand:

SOLUTION.

Make AB $=a$, AD $=b$, and EF $=c$, and DF $=x$; then will AF $=x+b$, and by reaſon of the Similitude of the Triangles BAF, EDE, and ECB,

FA : AB :: FD : DE

$$x+b : a :: x : \frac{ax}{x+b}$$

But now $\overline{DF}^2+\overline{DE}^2$ are $=\overline{EF}^2$, i. e. $\frac{aaxx}{xx+2bx+bb}+xx=cc$; or giving the ſame Denomination to all the Quantities on the left hand,

$$\frac{x^4+2bx^3+\begin{matrix}aaxx\\+bb\end{matrix}}{xx+2bx+bb}=cc,$$

and multiplying both Sides by $xx+2bx+bb$,

x^4

$$x^4 + 2bx^3 \begin{matrix} + aaxx \\ + bb \end{matrix} = ccxx + 2bccx + bbcc;$$

and tranſlating all the Quantities on the right hand with contrary Sines to the leſt,

$$x^4 + 2bx^3 \begin{matrix} + aaxx \\ + bb \\ - cc \end{matrix} - 2bccx - bbcc = 0.$$

Wherefore (putting *a* for 1 and *L. Rectum*) the central Rule will be $\frac{L}{2} + \frac{p^2}{8} + \frac{q}{2}$ (becauſe the Quantity q is negative in the Equation, for cc is greater than $aa + bb$) = AD; and $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} - \frac{r}{2}$ = DH: or, according to our Reduction,

$$\frac{a}{2} + \frac{bb}{2} + \frac{cc - aa - bb}{2}, \text{ i.e. } \frac{cc}{2} = \text{AD},$$

$$\text{and } \frac{2b}{4} + \frac{8b^3}{16} + \frac{2bcc - 2aab - 2b^3}{4} - \frac{2bcc}{2}.$$

$$\text{i. e. } \frac{b}{2} + \frac{b^3}{2} + \frac{bcc - aab - lcc}{2} - bcc, \text{ i. e.}$$

$$0 - \frac{bcc}{2} = \text{DH}.$$

Wherefore the *Geometrical Conſtruction* requires no other preparatory Determination by our Form, than of the Quantities $\frac{cc}{2}$ for AD, $\frac{bcc}{2}$ for DH to get the Center, and $bbcc$ to determine the Radius of the Circle; which are exhibited by *Fig.* 56. *N.* 2. *viz.* NP is $= cc$, RS $= bcc$, RV $= bbcc$; which are found by means of LM $= a$, LR $= b$, LN and MO $= c$, MQ = NP, and MT = RS. Having therefore deſcribed a Parabola, *N.* 3. and drawn its Diameter, (becauſe the Quantity p is in the Equation) lay off AD = $\frac{1}{2}$NP in it, and alſo $\frac{1}{2}$RS from D to H perpendicularly, and on the right hand (becauſe DH $= -\frac{bcc}{2}$;) and ſo you will have the Center H, thro' which having drawn KAI, ſo that AK be = to the Quantity $bbcc$, or S, *i. e.* RV, &c. a Circle deſcribed at the Interval HL, will cut the Parabola in M and N, and applying the Magnitude NO, it will be = DF, that of the Curtain ſought; upon which, *N.* 4. having laid down the Circuit of the horned Work by help of the given Lines AB and AD, you'll have the Line EF, of the Magnitude which was above ſuppoſed. Now if any one has a mind to do the ſame thing by *Baker's* Way; by laying down firſt the Diſtance $Ab = \frac{L}{2}$, and then making $bc = \frac{p^2}{8}$, and laſtly, putting cd for the Quantity $\frac{q}{2}$; he will fall upon the ſame Point D, and in like manner may expreſs the other Parts of the central Rule by the Diſtances De, ef, fg, and laſtly, laying off gh back again, he will fall upon the ſame Center H: But this is done with a great deal more Trouble and Labour to determine ſo many Quantities, and alſo is in more danger of erring, by cutting off ſo many Parts ſeparately, as Experience will ſhew; and thus we have, by a new Argument, ſhewn the Advantage of our Reduction.

Another Solution of the ſame Problem.

Things remaining as before, (only aſſuming the given Lines AB, AD, and EF, *Fig.* 56. *N.* 1. one half leſs, that the Scheme may take up leſs room) make $BE = x$, as the firſt or chief unknown Quantity; then will $BF = x + c$, and $\overline{BF}^2 = xx + 2cx + cc$: And ſince

$BE : BC = AD :: BF : AF$ a fourth, whence
$x : b :: x + c$

$AF = \frac{bx + bc}{x}$, and its Square $= \frac{bbxx + 2bbcx + bbcc}{xx}$.

Wherefore if this Square be ſubſtracted from the Square of BF, there will remain the Square of BA, *i. e.* $xx + 2cx + cc - \frac{bbxx - 2bbcx - bbcc}{xx} = aa$; *i. e.* all being reduced to the ſame Denomination,

$$x^4 + 2cx^3 \begin{array}{l} + ccxx \\ - bb \end{array} - 2bbcx - bbcc = aaxx;$$

or, according to the Forms of *Des Cartes* and *Baker*,

$$x^4 + 2cx^3 \begin{array}{l} + ccxx \\ - aa \\ - bb \end{array} - 2bbcx - bbcc = 0.$$

Therefore the Central Rule (putting again a for 1 and the *L. R.*) will be $\frac{L}{2} + \frac{p^2}{8} - \frac{q}{2} = AD$, and $\frac{p}{4} + \frac{p^3}{16} -$

—

$-\frac{pq}{4}-\frac{r}{2}=$ DH; or by our Reduction $\frac{a}{2}+\frac{cc}{2}-\frac{cc+aa+bb}{2}$ i. e. $a+\frac{bb}{2}=$ AD; and $\frac{c}{2}+\frac{c^3}{2}-\frac{c^3+aac+bbc-bbc}{2}$ i. e. $c-\frac{bbc}{2}=$ DH.

Geometrical Conſtruction. Having therefore deſcribed a Parabola (*Fig.* 56. *N.* 5.) and drawn the Diameter AY, make A 1 $=a$, and 1, 2, $=\frac{bb}{2}$; ſo you'll have the Point D; make moreover D3 or 2, 3 $=c$, and 3, 4 back again $=\frac{bbc}{2}$ (we here omit to expreſs the Geometrical Determination of theſe Quantities $\frac{bb}{2}$ and $\frac{bbc}{2}$, as being very eaſy) and you'll have the Point H, &c. and there will come out the Quantity ſought NO; which ſince it is equal to half BE *N.* 4. the Buſineſs will be done; which *Baker*'s Form will alſo give, exactly the ſame, but after a more tedious Proceſs.

PROBLEM IX.

In any Triangle ABC (the Scheme whereof, ſee N.1. Fig. 57.) ſuppoſe the Perpendicular AD, and the Differences between the leaſt Side, and the two others EC and FC, being given, to find all the three Sides; i. e. Chiefly the leaſt Side AB, which being found, the others will be ſo alſo.

SOLUTION.

Make AD $=a$, EC $=b$, and FC $=c$, AB $=x$; then will BC $=x+b$ and its Square will be $xx+2bx+bb$, and AC $=x+c$ and its Square $xx+2cx+cc$; and BD will be $=\sqrt{xx-aa}$, and DC $=\sqrt{xx+2cx+cc-aa}$. But $\overline{BC}^2$ may alſo be obtain'd otherwiſe, and ſo the Equation, if $\overline{BD}^2+\overline{DC}^2+2\,BD\times DC$ be added into one Sum, according to *Prop.* 4 *Lib.* 2. *Eucl.* viz.

$$2xx+2cx+cc-2aa+\sqrt{4x^4+8cx^3+\left.\begin{matrix}4cc\\-8aa\end{matrix}\right\}xx-8aacx+4a^4-4aacc}$$ will be $=xx+2bx+bb$,

[For DC $=\sqrt{xx-aa+2cx+cc}$, multiplied by BD $\sqrt{xx-aa}$, gives the Rectangle of the Segments

$\sqrt{x^4}$

$$\sqrt{x^4+2cx^3\begin{matrix}+ccxx\\-2aa\end{matrix}-2aacx+a^4-aacc}$$

and this doubled, *i. e.* multiplying by $\sqrt{4}$, gives the Quantity which is contain'd under the radical Sign in the Equation.]

Therefore turning all over on the left hand, which are before the Sign $\sqrt{}$, to the right hand, prefixing to them the contrary Signs, you'll have

$$\sqrt{4x^4+8cx^3\left.\begin{matrix}+4cc\\-8aa\end{matrix}\right\}xx-8aacx+4a^4-4aac}$$

$$=-xx\frac{+2b}{-2c}x+bb-cc+2aa,$$ and taking away the *Vinculum* on the left hand, and Squaring on the right,

$$4x^4+8cx^3\left.\begin{matrix}+4cc\\-8aa\end{matrix}\right\}xx-8aacx+4a^4-4aacc$$

$$=x^4\begin{matrix}+4cx^3\\-4b\end{matrix}\left.\begin{matrix}-4aa\\+2bb\\+6cc\\-8bc\end{matrix}\right\}xx\left.\begin{matrix}+4b^3\\+4c^3\\+8aab\\-8aac\\-4bbc\\-4bcc\end{matrix}\right\}x\begin{matrix}+4a\\+b^4\\+c^4\\+4aabb\\-4aacc\\-2bbcc\end{matrix}$$

and adding and ſubſtracting on both Sides, as much as can be,

$$3x^4+4cx^3-4aaxx$$
$$=-4bx^3\left.\begin{matrix}+2bb\\+2cc\\-8bc\end{matrix}\right\}xx\left.\begin{matrix}+4b^3\\+4c^3\\+8aab\\-4bbc\\-4bcc\end{matrix}\right\}x\begin{matrix}+b^4\\+c^4\\+4aabb\\-2bbcc\end{matrix}$$

and transferring all to the left,

$$3x^4\left.\begin{matrix}+4b\\+4c\end{matrix}\right\}x^3\left.\begin{matrix}-4aa\\-2bb\\-2cc\\+8bc\end{matrix}\right\}xx\left.\begin{matrix}-4b^3\\-4c^3\\-8aab\\+4bbc\\+4bcc\end{matrix}\right\}x\begin{matrix}-b^4\\-c^4\\-4aabb\\+2bbcc\end{matrix}=0$$

and dividing all by 3,

$$x^4\left.\begin{matrix}+\frac{4}{3}b\\+\frac{4}{3}c\end{matrix}\right\}x^3\left.\begin{matrix}-\frac{4}{3}aa\\-\frac{2}{3}bb\\-\frac{2}{3}cc\\+\frac{2}{3}bc\end{matrix}\right\}xx\left.\begin{matrix}-\frac{4}{3}b^3\\-\frac{4}{3}c^3\\-2\frac{2}{3}aab\\+\frac{4}{3}bbc\\+\frac{4}{3}bcc\end{matrix}\right\}x\begin{matrix}-\frac{1}{3}b^4\\-\frac{1}{3}c^4\\-\frac{4}{3}aabb\\+\frac{2}{3}bbcc\end{matrix}=0.$$

Note, *I ſought this Equation alſo after two other ways ; 1. By a Compariſon of the* $\overline{\mathrm{AC}}$ *with* $\overline{\mathrm{AB}}^2+\overline{\mathrm{BC}}^2$, after

2CBD *thence ſubſtracted, (according to* Prop. 13. Lib. 2. Eucl. which is the 46. Lib. 1. Math. Enucl.) and I form'd the ſame *with the preſent.* 2. *By putting in the beginning* y *for* x + b, *and* z *for* x + c, *and going on after the former methods, till I got this Equation,*

$$\left.\begin{matrix} x^4 - 2x^2 \\ + 4aa \end{matrix}\right\} yy \left.\begin{matrix} - 2x^2 \\ - 2y^2 \end{matrix}\right\} zz \begin{matrix} + y^4 \\ + z^4 \end{matrix} = 0.$$

in which, when afterwards I ſubſtituted the Values anſwering the Quantities yy *and* zz, &c. *this ſame laſt Equation came out a little eaſier, but* (N. B.) *with all the contrary Signs.*

Now to form the Central Rule, and thence make the Geometrical Conſtruction, we muſt determine firſt, each of the Quantities p, q, r and S, that we may know whether they are negative or poſitive; and you'll find (*N.* 2. *Fig.* 57.) $p =$ G2 poſitive, $q =$ H2 negative, and K4 $= S$ alſo negative; and that by help of the Quantities LM $= b$ or 1, MN and LO $= a$, OP $= aa$, MQ and LR $= c$, RS $= cc$, and alſo LT $= cc$, TV and LX $= c^3$, XY $= c^4$. Wherefore the Form of the laſt Equation will be like this,

$x^4 + px^3 - qxx - rx - S = 0$, and ſo the Central Rule (taking here b for 1 and the *L. R.*) will be $\frac{L}{2} + \frac{p^2}{8} + \frac{q}{2} =$ AD, and $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} - \frac{r}{2} =$ DH.

Wherefore, having now deſcribed a Parabola (as may be ſeen *N.* 4.) and found the Diameter AY lay off in it firſt A$b = \frac{1}{2}$ LM (*N.* 2.) and then $bc = \mathfrak{DP}$ (*N.* 3.) i. e. $\frac{p^2}{8}$; and thirdly cD $= \frac{q}{2}$, i. e. $\frac{1}{2}$ H4 (*N.* 2.) moreover from D to e put off $\mathfrak{PB}$ (*N.* 3.) $= \frac{p}{4}$, and from e to f put off $\mathfrak{DR} = \frac{p^3}{16}$, and from f to g put off $\mathfrak{CF} = \frac{pq}{4}$; and from g backwards to h put off half the Quantity r, or I5 (*N.* 2.) and having done the reſt as uſual, you'll have NO, the Side required of the Angle to be deſcrib'd; the Deſcription whereof will be now eaſy (*N.* 5.) having all the three Sides known. This may ſerve for an Examen, if having deſcribed a Semicircle AGB upon AB = NO, you apply the given Line AD, in it, and from B thro' D draw BDC: then at the Interval AB having deſcribed the Arches AE and BF, add the given

Line EC to BE, for thus having joined the Points A and C, FC ought to be equal to FC before given.

SCHOLIUM.

WE have here omitted our Reduction, becauſe it would be too tedious, and would expreſs the Quantities AD and DH (eſpecially the latter) in Terms two prolix. For AD would be $= \frac{b}{2} + \frac{5}{9}bb + \frac{5}{9}cc - \frac{8}{9}bc + \frac{2}{3}aa$ (*viz.* becauſe $\frac{p^2}{8}$ is found $= \frac{2}{9}bb + \frac{4}{9}bc + \frac{8}{9}cc$, and $\frac{q}{2}$ taken in it ſelf $= \frac{2}{3}aa - \frac{1}{3}bb - \frac{1}{3}cc + \frac{4}{3}cc$; but here [where by virtue of the Central Rule it is taken poſitively, when it is in it ſelf negative] under contrary Signs it is $= \frac{2}{3}aa + \frac{1}{3}$ or $\frac{3}{9}bb + \frac{1}{3}$ or $\frac{3}{9}cc - \frac{4}{3}$ or $\frac{12}{9}bc$) or yet more contractedly (becauſe b is Unity) AD $= \frac{1}{2}b + \frac{5}{9}b$ (i.e. $1\frac{1}{18}b$) $+ \frac{5}{9}cc - \frac{8}{9}c + \frac{2}{3}aa$; which Parts may be expreſſed without any great difficulty in the Diameter AY, by its Portions A1, 1, 2; 2, 3; 3D: But the other Quantity DH, by means whereof we got the Center H, would alſo have ſome tediouſneſs, as becauſe

$\frac{p}{4}$ would be $= \frac{b+c}{3}$

$$\frac{p^2}{16} = \frac{4b^3 + 12b^2c + 12bcc + 4c^3}{27}$$

$$\frac{pq}{4} = \frac{6bcc + 6bbc - 4aab - 4aac - 2b^3 - 2c^3}{9}$$

and $\frac{r}{2} = \frac{2bbc + 2bcc - 4aab - 2b^3 - 2c}{3}$

or $\frac{6bbc + 6bcc - 12aab - 6b^3 - 6b^3}{9}$

If from the Quantities $\frac{pq}{4}$ and $\frac{r}{2}$ (ſince this latter is to b ſubſtracted, and ſo left, as it is, under the Sign —; bu the other, alſo negative in it ſelf, but here poſitively ex- preſs'd in the Central Rule, muſt have all the contrary Signs you take thoſe which deſtroy one another, and add the reſt t the two former Quantities, then will

$$\frac{b+c}{3} - \frac{8aab + 4aac + 4bbc + 4bcc}{9} - \frac{8b^3 - 8c^3}{27} = \text{DH}$$

or a litle more contracted (becauſe b is 1) $\frac{b+c}{3}$

$$-\frac{8aa + 4aac + 4c + 4cc}{9} - \frac{8b^3 - 8c^3}{27} = \text{DH}.$$

But now if any one has a mind to illuſtrate this by a numeral Example, and try the Truth, &c. *of the Quantities found, they may make,* e. g. $a = 12$, $b = 1$, and $c = 2$; *and they will eaſily find that in the laſt Equation the Quantity* p *will be* 4, and $q, -190$, $r, -388$, $S, -195$: Secondly, *in the Central Rule of* Baker *they'll find* $\frac{L}{2} = \frac{1}{2}$, $\frac{p^2}{8} = 2$, and $\frac{q}{2} = 95$, *and ſo the whole Line* AD $= 97\frac{1}{2}$; *and further* $\frac{p}{4} = 1$, $\frac{p^2}{16} = 4$, $\frac{pq}{4} = 190$, and $\frac{-r}{2} = -194$; *and ſo the whole Line* DH $= 195 - 194$, i. e. $= 1$. Thirdly, *likewiſe in our Reduction* (if we proceed by each Part correſponding to *Baker's*) $\frac{b}{2} = \frac{1}{2}$, $\frac{2}{9}bb + \frac{4}{9}bc + \frac{2}{9}bc = 2$, and $\frac{1}{3}aa + \frac{4}{3}bb + \frac{4}{3}cc - 4bc = 95$. *The Sum for* AD $97\frac{1}{2}$; *but further* $\frac{b+c}{3} = 1$, $\frac{4b^3 + 12bbc + 12bcc + 4c^3}{27} = 4$;

$$\frac{6bcc + 6bbc - 4aab - 2b^3 - 2c^3}{9} = 190;$$

and $\frac{2bbc + 2bcc - 4aab - 2b^3 - 2c^3}{3} = 194$ *to be ſubſtracted; and ſo the Sum for* DH $= 195 - 194 = 1$. *Which ſame Quantities will* (fourthly) *come out, if the Quantities* AD *and* DH *contracted, as they are expreſſed in Letters above, be reſolved into Numbers.*

PROBLEM X.

TO build a Fort on the given Polygons EAF (*ſee Fig.* 58. N. 1.) *whoſe capital Line* AB *ſhall be equal to the Aggregate of the Gorge and Flank, and the Squares of theſe added together ſhall be equal to the Square of the given Line* GH, *and the Solid made by the Multiplication of the Square of the Flank by the Gorge, ſhall be equal to the Cube of the given Line* IK.

SOLUTION.

Make the Gorge AC $= z$, whoſe Square zz ſubſtracte from bb the Square of the give Line b, will leave the Squa of the Flank DC $= bb - zz$. Now this Square bein multiplied by the Gorge AC or z will give $bbz - z^3 = g$ the Cube of the given Line IK; and adding to both Sid z^3; and ſubſtracting bbz, $0 = z^3 * - bbz + g^3$.

Therefore if we take g or IK for 1 and L^3, g^3 will be $=$ the Line g, and

The Central Rule will be: $\frac{L}{2} + \frac{q}{2} =$ AD.

and $\frac{r}{2} =$ DH.

i. e. according to our Reduction,

$\frac{g}{2} + \frac{bb}{2} =$ AD and $\frac{g}{2} =$ DH.

Geometrical Conſtruction. Having deſcribed a Parabol (*N.* 2. *Fig.* 58.) make in its Axis A1 $= \frac{1}{2}$ IK, and 1, 2, o 1 D $= \frac{1}{2}$ MN (*N.* 1.) and DH $= \frac{1}{2}$ IK. Then having de ſcribed a Circle from H, and found the affirmative Root NC upon the given Angle EAF (*N.* 3.) make AC $=$ NO, an having erected the Perpendicular CD divide it by AD $=$ GI (*N.* 1.) and make AB $=$ AC $+$ CD; and the Fort will b drawn.

PROBLEM XI.

IN a right-angled Triangle ABC (*which we denote by* N. *Fig.* 59.) *having given the greater Leg* AC, *and made th leſs Leg* AB $=$ *to the Segment* CE, *which is cut off from th Baſe* BC *by a Perpendicular let fall from the right Angle* A *to find theſe Lines* AB *or* CE, *and conſequently the whol Triangle.*

SOLUTION.

Make AC $= a$, and CE or AB $= x$. Then you'll hav $aa - xx = \square$ of AE. And becauſe the Triangles BEA and CAE are ſimilar, you'll have

as AC to CE ſo BA to AE

$$a : x :: x : \frac{xx}{a}$$

And

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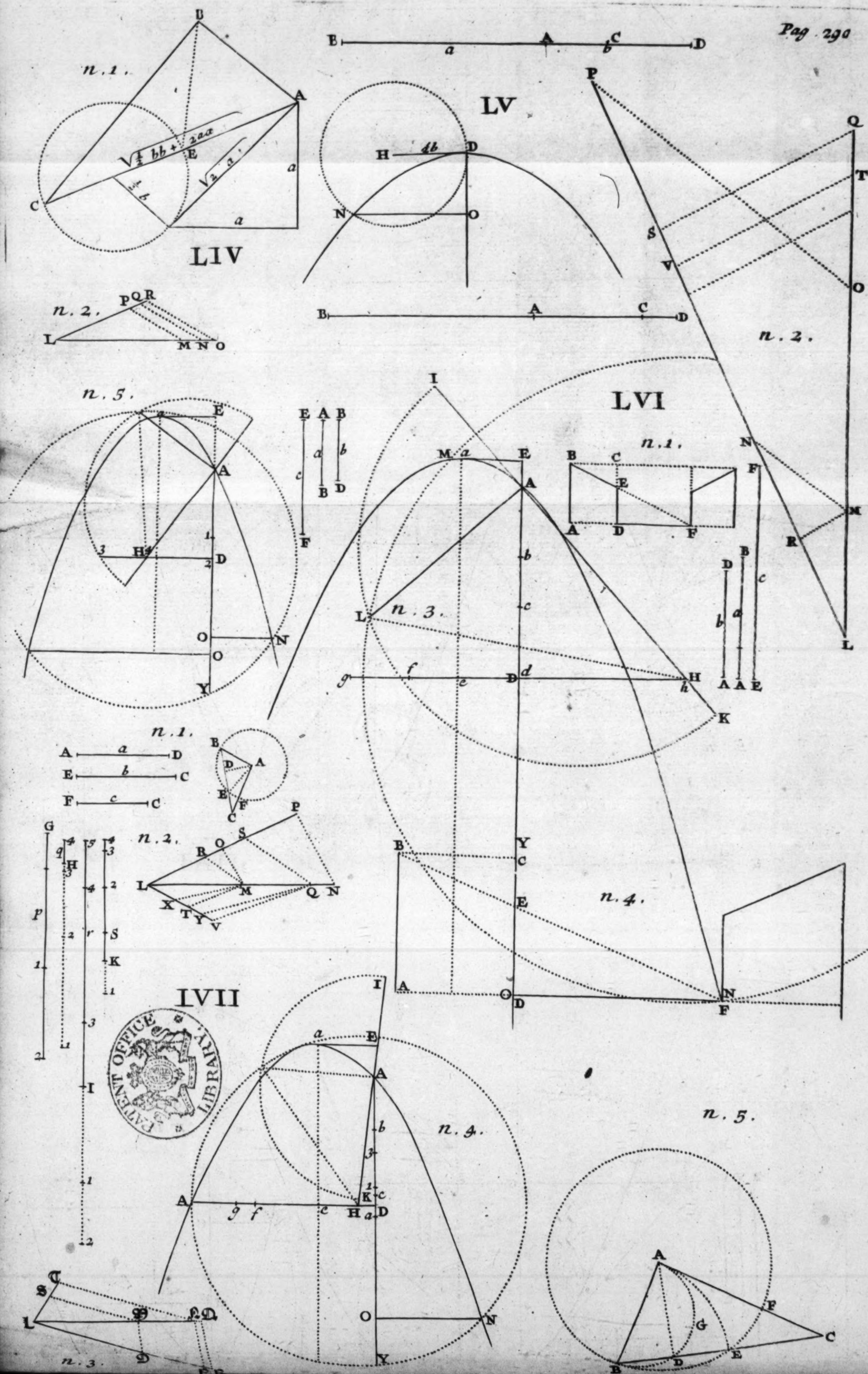
55	281
56	282
57	285



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And ſo, □ of

$\frac{x^4}{aa} = aa -$

$x^4 = a^4 - a$

tes and *Baker*, by

$x^4 * + aax$

$x^4 * + qxx$

Therefore (tak

will be $\frac{L}{2} - \frac{q}{2}$

and $\frac{r}{2}$ i. e. $o =$

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GF $= \frac{1}{2} a$, there
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Quantity *x* ſought

And ſo, □ of AE $= \frac{x^4}{aa}$. Therefore

$\frac{x^4}{aa} = aa - xx$; and multpl. by aa,

$x^4 = a^4 - aaxx$; or, according to the Form of *des Car-tes* and *Baker*, by tranſpoſing all to the left,

$x^4 * + aaxx * - a^4 = o$, i. e.

$x^4 * + qxx * - S = o$.

Therefore (taking a for 1 and the *L. R.*) the Central Rule will be $\frac{L}{2} - \frac{q}{2}$ i. e. $o =$ AD

and $\frac{r}{2}$ i. e. $o =$ DH; and ſo H falls on the Vertex A.

Geometrical Conſtruction. Since a is aſſumed for Unity and L, the Quantity S alſo and *Latus Rectum*, i. e. AI and AK, and conſequently the Mean Proportional AL and the Radius HL will be = to the given Side AC; and ſo at that interval having deſcrib'd a Circle, thro' the Parabola rightly delineated, you'll have NO the Value of the Quan-tity x, *i. e.* of the Leg AB. Having drawn therefore NA, which is equal to AC by Conſtruction, if you draw to it the Perpendicular AB cutting NO produc'd to B, you'll have the Triangle ſought ABC, and AB (which will be a Sign of true Solution) will be found = NO or CE.

Another Conſtruction. Since in the Equation above found here is neither x^3 nor x, it may be look'd upon as a Quadra-ick, and conſtructed after the ſame way, as ſeveral others ike it among the Examples in the Conſtruction of Equations, iz. becauſe $x^4 = - aaxx + a^4$; according to Caſe 2. of ffected Quadraticks,

xx will be $= -\frac{1}{2}aa + \sqrt{\frac{1}{4}a^4 + a^4}$ i. e. $+ \sqrt{\frac{5}{4}a^4}$,

nd $x = \sqrt{-\frac{1}{2}aa + \sqrt{\frac{5}{4}a^4}}$ i. e. $\sqrt{-\frac{1}{2}a + \sqrt{\frac{5}{4}a}}$

Wherefore (*N.* 3. *Fig.* 59.) if AC be made $= a$, and CD $\frac{5}{4}a$ he Mean Proportional CG will be $= \sqrt{\frac{5}{4}a}$, and taking hence GF $= \frac{1}{2}a$, there will remain FC $= \sqrt{\frac{5}{4}a} - \frac{1}{2}a$: And now etween this or CH equal to it, and Unity AC, having found nother Mean Proportional CE, it will be the Value of the Quantity x ſought = NO (*N.* 2.)

PROBLEM XII.

IN a right-angled Triangle ABC (*Fig.* 60. *N.* 1.) *there is given the Perpendicular* BA, *a Segment* BD *of the Hypothenuſe, and* EC *a Segment of the Baſe, from* C *to the Perpendicular* DE *let fall from the end of the Segment* BD; *to find* AEDC, *and conſequently the Baſe* AC *and the Hypothenuſe* BC, *and ſo the whole Triangle.*

SOLUTION.

Make AB $= a$, BD $= b$, EC $= c$, and DC $= x$; which being given, the reſt cannot be wanting: Therefore $xx - cc = \square$ of DE. But the ſame $\square$ of DE may be had, if by ſaying

$$\mathrm{BC} : \mathrm{BA} :: \mathrm{DC} : \mathrm{DE}$$

$$b + x : a :: x : \frac{ax}{b+x}$$

And then ſquare the Quantity DE, the Square will be

$$\frac{aaxx}{xx + 2bx + bb} = xx - cc;$$

and multiplying both Sides by $xx + 2bx + bb$,

$$aaxx = x^4 + 2bx^3 + \begin{array}{l} bbxx \\ -cc \end{array} - 2bccx - bbcc;$$

and ſubſtracting alſo $aaxx$,

$$x^4 + 2bx^3 + \begin{array}{l} bbxx \\ -aa \\ -cc \end{array} - 2bccx - bbcc = 0.$$

Which Equation (ſince bb is greater than $aa + cc$) wil give this Form according to *Baker*;

$$x^4 + px^3 + qxx - r - S = 0.$$

Wherefore the Central Rule will be (taking a for Uni and L)

$$\frac{\mathrm{L}}{2} + \frac{p^2}{8} - \frac{q}{2} = \mathrm{AD}$$

$$\text{and } \frac{p}{4} + \frac{p^3}{16} - \frac{pq}{4} - \frac{r}{2} = \mathrm{DH};$$

or according to our Reduction,

$$\frac{a}{2} + \frac{bb}{2} - \frac{bb + aa + cc}{2}, \text{ i. e. } a + \frac{cc}{2} = \mathrm{AD};$$

and $\frac{b}{2}+\frac{b^3}{2}-\frac{b^3+aab+bcc\,bcc}{2}$, i. e.

$$b-\frac{bcc}{2}=\mathrm{DH}.$$

Geometrical Conſtruction; for the Foundation of which there are found (*N.* 2.) and partly aſſumed, LM $= a$, MN and LO $= c$, OP and MQ $= cc$, LR $= \frac{1}{2} b$, RS $= \frac{1}{2} bcc$; (*N.* 3.) $\mathfrak{LM} = a$, $\mathfrak{LN}$ and $\mathfrak{LO} = b$, $\mathfrak{LP} = bb$, $\mathfrak{LQ} = cc$, $\mathfrak{LR} = bbcc$. Having therefore delineated the Parabola, (*N.* 4.) and found its Diameter, lay off LM in it, from A to 1, and $\frac{1}{2}$ OP from 1 to 2 or D; and moreover BD (*N.* 1.) from D to 3, and RS back again from 3 to 4 or H, the Center ſought. Then having made AK $= \mathfrak{LR}$, and done the reſt as uſual, you'll find NO, or the Quantity DC, for the Value of x; to which if you add BD, and, having upon the whole Line BC deſcrib'd a Semicircle, apply in it BA, and you'll have alſo AC and the whole Triangle ſought, and you'll find EC (letting fall a Perpendicular from D) equal to that Quantity which had the ſame Name, *N.* 1.

If any one had rather immediately conſtruct it by *Baker*'s Form, there muſt be made ſeveral Preparatives and Aſſumtives, *viz.* (*N.* 5.) $\mathfrak{LP} = bb$, $\mathfrak{LM} = a$ or aa, $\mathfrak{MQ} = cc$, or OP (*N.* 2.) that you may have $\mathfrak{QP} = q$: Moreover PQ $= a$, PR $= \frac{1}{2} p$ or BD (*N.* 1.) QS $= a \frac{1}{4} p$ or $\frac{1}{2}$ BD; that RT may be $\frac{pp}{8}$ and alſo PV $=$ to it, that PX may be $\frac{p^3}{16}$, and laſtly PY $= \frac{1}{2} q$, that PZ may be $\frac{pq}{4}$. Things being thus prepar'd, if (*N.* 4.) Ab be made $= \frac{1}{2}$ AB (*N.* 1.) $bc =$ RT, and cd be ſet off backwards $= \frac{1}{2}\mathfrak{PQ}$, we light on the Point D; and, if we make D$e = \frac{1}{2}$ BD, $ef =$ PX (*N.* 5. which interval was too big to be repreſented in the Paper) and from f put fg $=$ PZ backwards, and from g, $gh = 2$RS; we ſhall light on the point H, &c. Which of theſe two methods is the ſhorteſt and fitteſt for practice, any one, never ſo little experienc'd, may here ſee: and firſt Learners may take notice if they would conſtruct by *Baker*'s Form, in the Diagrams *N.* 2. *N.* 5. and the like, they muſt take care to make the Angles $\mathfrak{PLN}$, SPT pretty large; which we have here repreſented the leſs, to ſave Charges in cutting on Copper.

PROBLEM XIII.

THE Diameter of a Circle CD (Fig. 61. N. 1.) *and the Line* BG, *which falls on it perpendicularly, being given; to find the Point* A : *from which a right Line* AC *being drawn, ſhall ſo cut the Line* BG *in* F, *that* AF, FG, GD *ſhall be three continual Proportionals.*

SOLUTION.

If CF be found, the Point A will be alſo had, and the Section of the Line BG will be made. Make therefore CF $= x$, and (becauſe the Perpendicular BG is given, there will alſo be given the Segments of the Diameter CG and GD) make GD $= b$, and CG $= c$; then will BG $= \sqrt{bc}$, and CD $= b + c$. Since therefore the Triangles CAD and CGF are right-angled, and have the Angle at C common, they will be ſimilar.

Therefore, as CF : CG :: CD : CA

$$x : c :: b + c : \frac{bc + cc}{x}$$

Therefore AF $= \frac{bc + cc}{x} - x$, and FG $= \sqrt{xx - cc}$.

But by the Hypotheſis,

as AF : FG :: FG : GD

$$\frac{bc + cc - xx}{x} : \sqrt{xx - cc} :: \sqrt{xx - cc} : b.$$

Therefore the Rectangle of the Extremes will be = to that of the Means,

$$\frac{bbc + bcc - bxx}{x} = xx - cc;$$

and multiplying by x,

$$bbc + bcc - bxx = x^3 - ccx;$$

and tranſpoſing all to the left,

$x^3 + bxx - ccx - bbc - bcc = 0$; *i.e.* by the *Carteſian* Form, $x^3 + pxx - qx - r = 0$.

Therefore (taking b for 1, and *L.*) the Central Rule will be

$$\frac{L}{2} + \frac{p^2}{8} + \frac{q}{2} = AD$$

and $\frac{p}{4} + \frac{p^3}{16} + \frac{pq}{4} - \frac{r}{2} = DH$,

or, according to our Reduction,

$\frac{b}{2} + \frac{bb}{8} + \frac{cc}{2}$, i.e. $\frac{5b}{8} + \frac{cc}{2} = AD$, and $\frac{b}{4}+\frac{b^2}{16}+\frac{cc}{4}-\frac{c-cc}{2}$, i. e. $\frac{5b}{16}-\frac{cc}{4}-\frac{c}{2}=$ DH.

Geometrical Conſtruction. Having deſcribed a Semicircle upon the given Line CD, (*N.* 2.) and apply'd in it the given Perpendicular BG, as the Figure ſhews, you'll have the Segments of the Diameter GD $= b$, and to the Quantity p in *Baker*'s Form, and CG $= c$, which (*N.* 3. where LM $= b$, LO and MN $= c$) will give OP $= cc$, and to the Quantity q. Wherefore having deſcribed a Parabola (*N.* 4.) and made the Line VZ $= 2\frac{1}{2}b$, and cut off the fourth Part of XZ, and the eighth of YZ (whereof the one will be $= \frac{2\frac{1}{2}b}{8}$, viz. $\frac{5b}{16}$, and the other to $\frac{5b}{8}$) if you lay off A1 $=$ XZ in the Diameter of the Parabola AY, and moreover 1, 2, or 1D $=$ to half OP, (*N.* 3.) and D3 $=$ YZ, and 3, 4 $= \frac{1}{4}$OP, as alſo 4, 5 $= \frac{1}{2}$CG, (*N.* 2.) you'll have the Center H; and having deſcribed a Circle at the Interval HA, the Root NO muſt be transferr'd from (*N.* 2.) C to F, and continued to A the Point ſought. In *Baker*'s Form (becauſe the Quantity p is $= b$ or 1) $\frac{p^2}{8}$ is $= \frac{b}{8}$, and $\frac{b^2}{16} = \frac{b}{16}$, and the Quantity q or cc $=$ OP (*N.* 3.) therefore in the Diameter of the Parabola, make A$b = \frac{1}{2}$GD, and $bc = \frac{1}{8}$GD. (*N.* 2.) and laſtly, cD $= \frac{1}{2}$OP, (*N.* 3.) and you'll have the Point D the ſame as before. Make moreover D$e = \frac{1}{4}$GD, and $cf = \frac{1}{16}$CG, and then $fg = \frac{1}{4}$OP, and laſtly, $gh = \frac{1}{2}$GD, and you'll have the ſame Center H, and the Co-incidence of the Parts in both Forms will be pleaſant to obſerve; which otherwiſe ſeldom happens.

Other Solutions of the ſame Problem.

Charles Renaldinus, from whoſe Treatiſe *de Reſol. & Compoſ. Math. lib.* 2. we have the preſent Problem, proceeds to ſolve it in another way, changing it plainly into another Problem; *viz.* He obſerves, (1.) That the Angles FAD (ſee *N.* 1. of our 61 *Fig.*) and FGD being both right ones on the ſame common Baſe FD, are in Circle. Hence he infers, (2.) By virtue of the *Coroll.* of the 26. *Prop.* 3. *Eucl.*) that the Rectangles DCG and ACF are equal; and conſequently CD, CA, FC, and CG are four continued Proportionals. Then he obſerves, (3.) That GD is the Exceſs of the firſt of theſe Proportionals above the fourth CG, and AF is the Exceſs of the ſecond AC above the third CF: And ſo, ſince, (4.) the Rectangle of AF and GD is = to the Square of the mean Proportional FG (for AF, FG, GD, are ſuppoſed to be continual Proportionals) and this □ of FG is the Exceſs, by which the Square of the third CF exceeds the Square of the fourth CG: Now the preſent Problem will be, (5.) reduced to this other: *Two right Lines* (CD *and* CG) *being given, to find two ſuch mean Proportionals* (AC *and* FC) *that the Rectangle of the Exceſs of the firſt above the fourth* (viz. *of* FA *into* GD) *ſhall be equal to the Exceſs, by which the Square of the third* (FC) *exceeds the Square of the fourth* (CG) viz. *by the Square* FG.

Wherefore, inſtead of the former, he ſolves this latter Problem, putting b for CG, and c for GD; ſo that the firſt of the given Lines CD ſhall be $=b+c$, and the other GD $=b$; calling the firſt mean Proportional AC, x; and thence denominating the latter $\frac{bb+bc}{x}$, (*viz.* multiplying the fourth by the firſt, and dividing the Product by the ſecond) and moreover he finds the Exceſs of the firſt $(b+c)$ above the fourth (b) to be c, and the Exceſs of the ſecond (x) above the third $\left(\frac{bb+bc}{x}\right)$ to be $x-\frac{bb+bc}{x}$, i.e. $\frac{xx-bb+bc}{x}$; ſo that the ▭ of theſe two Exceſſes is $\frac{cxx-bbc+bcc}{x}$: and becauſe the □ of the third FC is $=\frac{b^4+2b^3c+bbcc}{xx}$, having ſubſtracted $bb=$ □ of GC, there is given the □ of FG$=\frac{b^4+2b^3c+bb}{x}-bb=$ ▭ of the Exceſſes we juſt now found. So that now you'll have the Equation

$$\frac{b^4+2b^3c+bbcc-bbxx}{xx}=\frac{cxx-bbc+bcc}{x}, \text{ \&c.}$$

We alſo endeavour'd to find another Solution, by finding an Equation from the Line FD (*Fig.* 61. *N.* 1.) as which might be twice obtain'd by means of the two right-angled Triangles FAD and FGD, ſince it is the Hypothenuſe of both. But here, beſides the former Denominations of our Solution, we muſt firſt give a Denomination to the Line AD, by making

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59	290
60	292

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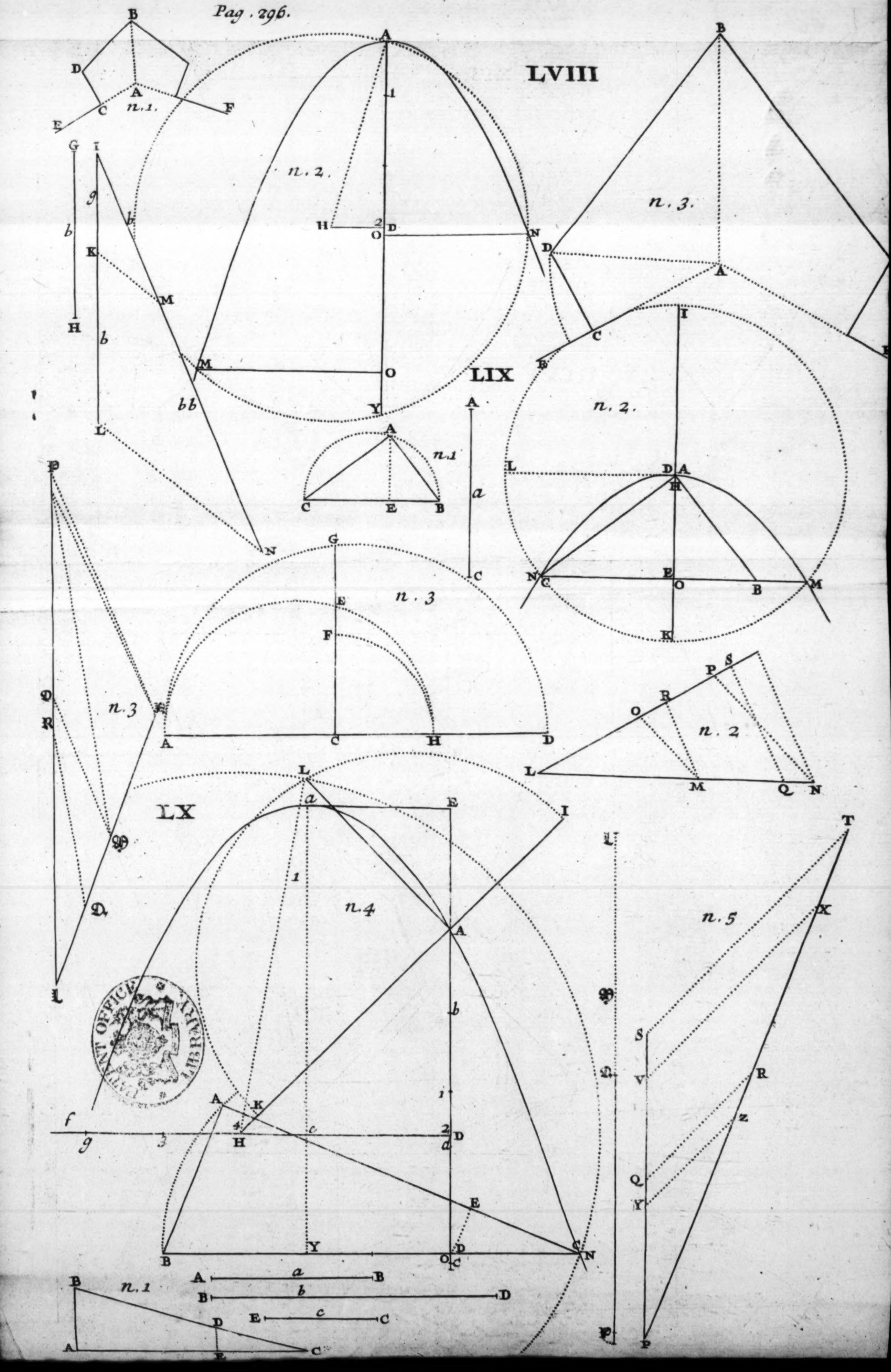

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$$CF : FG :: CD : AD$$

$$x : \sqrt{xx - cc} :: b + c, \text{ or } a : a\sqrt{\frac{xx - cc}{x}}, \text{ \&c.}$$

But whoſoever ſhall proſecute this Solution of ours, or that of *Renaldinus* to the end, will find much more Labour and Difficulty in either, than in the firſt we have given.

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APPENDIX.

THE Invention of the ſpecial Central Rule for the Caſe of Problem 1. Of Cubick Equations, &c. *which may ſerve for an Example for all the other ſpecial ones which belong to our Synopſis,* p. 354. *(and from theſe ſpecial ones) to find a general one.*

In *Fig.* 47. *N.* 4. make AD $= b$, DH $= d$; and ſo we ſhall have the □ of the Radius HA $= bb + dd$. But this □ of HA or HN, may be had alſo otherwiſe: by putting,

2. For the Quantity NO, as ſought, the Letter x, and by inferring from the known Property of the Parabola,

$$L : NO :: NO : AO$$

$$L : x :: x : \frac{xx}{L}$$

and ſubſtracting AD $= b$ from AO, you'll have DO or PN $= \frac{xx}{L} - b$; whoſe □ is $\frac{x^4}{L^2} - \frac{2bxx}{L} + bb$.

But you alſo have PH $=$ DH $-$ DP, or NO, *i. e.* $d - x$; whoſe □ is therefore $= dd - 2dx + xx$.

Wherefore adding the Squares of PN and PH, you'll have the □ of HN $= \frac{x^4}{L^2} - \frac{2bxx}{L} + xx - 2dx + bb + dd$.

Wherefore the Equation will now be readily had:

$$(\square \text{ of HN}) \ldots\ldots\ldots\ldots\ldots (\square \text{ of HA})$$

$$\frac{x^4}{L^2} - \frac{2bxx}{L} + xx - 2dx + bb + dd = bb + dd;$$

and taking from both Sides $bb + dd$,

$$\frac{x^4}{L^2} - \frac{2bxx}{L} + xx - 2dx = 0.$$

L^2 and multiplying all by $L^2 z$,

and alſo dividing by x,

x^3 $$x^4{}_{*} - 2bLx - 2L^2d = 0.$$
$$+ L^2x$$

3. And now farther comparing this Equation with a Form like ours in *Probl.* 1. *viz.* with this,

$$x^3{}_{**} - r = 0;$$

It is manifeſt, ſince in ours alſo the third Term, *viz.* q is wanting, that the correſpondent one to it in the former $-2bL + L^2$ is equivalent to 0; and adding to both Sides $2bL$, L2 will be made $= 2bL$; and dividing by $2L$, $\frac{L^2}{2L}$ will $= b$, or AD. In like manner, ſince $-r$ in our Form anſwers to the Quantity $2L^2d$ in the former, $2L^2d$ will $= r$; and, dividing by $2L^2$, DH or d will $= \frac{r}{2L^2}$: Which is the other Member of the Central Rule to be found.

N.B. *The Analytick Art has this particularly to be admired in it, that it finds its own Rules by an Analyſis: Whereof we have here an evident Example, and ſeveral others in the Reſolution of affected Quadraticks, and the following.*

II.

The Invention of the Central Rule, in the Caſe of Fig. 11. *and the like.*

1. $\square$ of HA $= bb + dd$, as above.

2. Putting x for NO, as ſought, we may infer from a new Property of the Parabola, which we have demonſtrated, *Prop.* 6. *lib.* 2.

As L to NO, ſo OR to AO, *i.e.* (putting a for BA or FO given; that NO — OF, *i.e.* NF or OR may be $= x - a$) as L to x, ſo is $x - a$ to $\frac{xx - ax}{L} =$ AO.

Therefore having ſubſtracted AD, *i. e.* b from AO, you'll have DO or PH $= \frac{xx - ax}{L} - b$; whoſe □ is

$$\frac{x^4 - 2ax^3 + axx}{L^2} - \frac{2bxx}{L} + \frac{2abx}{L} + bb.$$

But PN alſo is $=$ NO $-$ PO, or HD is $= x - d$, whoſe □ $= x^2 - 2dx + dd$.

Therefore having added the Squares of PH and PN together, there will come out □ of HN

$$= \frac{x^4 - 2ax^3 + aaxx}{L^2} - \frac{2bxx + 2abx}{L} + xx - 2dx + bb + dd$$

$=$ □ of HA, *i. e.* $bb + dd$; and taking away from both Sides $bb + dd$,

$$\frac{x^4 - 2ax^3 + aaxx}{L^2} - \frac{2bxx + 2abx}{L} + xx - 2dx = 0;$$

and multiplying every where by L^2, and dividing by x,

$$x^3 - 2ax^2 + aax - 2Lbx + 2Lab + L^2x - 2L^2d = 0.$$

3. And now comparing this Equation with another Form, which may be like an Equation ariſing from the Solution of ſome Problem, *e. g.* with this $x^3 - pxx + qx - r = 0$; to this you'll have $=$ this other,

$$x^3 - 2axx + \left.\begin{matrix} aa \\ -2Lb \\ +L^2 \end{matrix}\right\} x \begin{matrix} +2Lab \\ -2L^2d. \end{matrix}$$

4. Wherefore, becauſe in theſe equal Forms, *Firſt*, $2a$ is $= p$, a will be $= \frac{p}{2}$, *i. e.* the Line BA. *Secondly*, Becauſe

$$aa \left(\text{or } \frac{p^2}{4}\right) + L^2 - 2Lb = q;$$

Therefore $2Lb = \frac{p^2}{4} + L^2 - q$, and dividing by $2L$,

$$b = \frac{L^2}{2L} \left(\text{or } \frac{L}{2}\right) + \frac{p^2}{8L} - \frac{q}{2L} = \text{AD}.$$

Thirdly, Becauſe $2abL - 2dL^2 = r$, i. e.

$2abL - r = 2dL^2$; therefore dividing by $2L^2$, d will be $= \frac{2abL}{2L^2} - \frac{r}{2L^2}$, i. e.

$d = \frac{ab}{L} - \frac{r}{2L^2}$, *i. e.* reſolving $\frac{ab}{L}$ into equivalent Terms expreſſed by p and q,

d

$d=\frac{p}{4}+\frac{p^3}{16L^2}-\frac{pq}{4L^2}-\frac{r}{2L^2}$; which is the other Member of the Central Rule to be found.

viz. a is $=\frac{p}{2}$; $b=\frac{L}{2}+\frac{p^2}{8L}-\frac{q}{2L}$

Therefore ab will be $=\frac{Lp}{4}+\frac{p^3}{16L}-\frac{pq}{4L}$

Therefore $\frac{ab}{L}=\frac{p}{4}+\frac{p^3}{16L^2}-\frac{pq}{4L^2}$.

FINIS.

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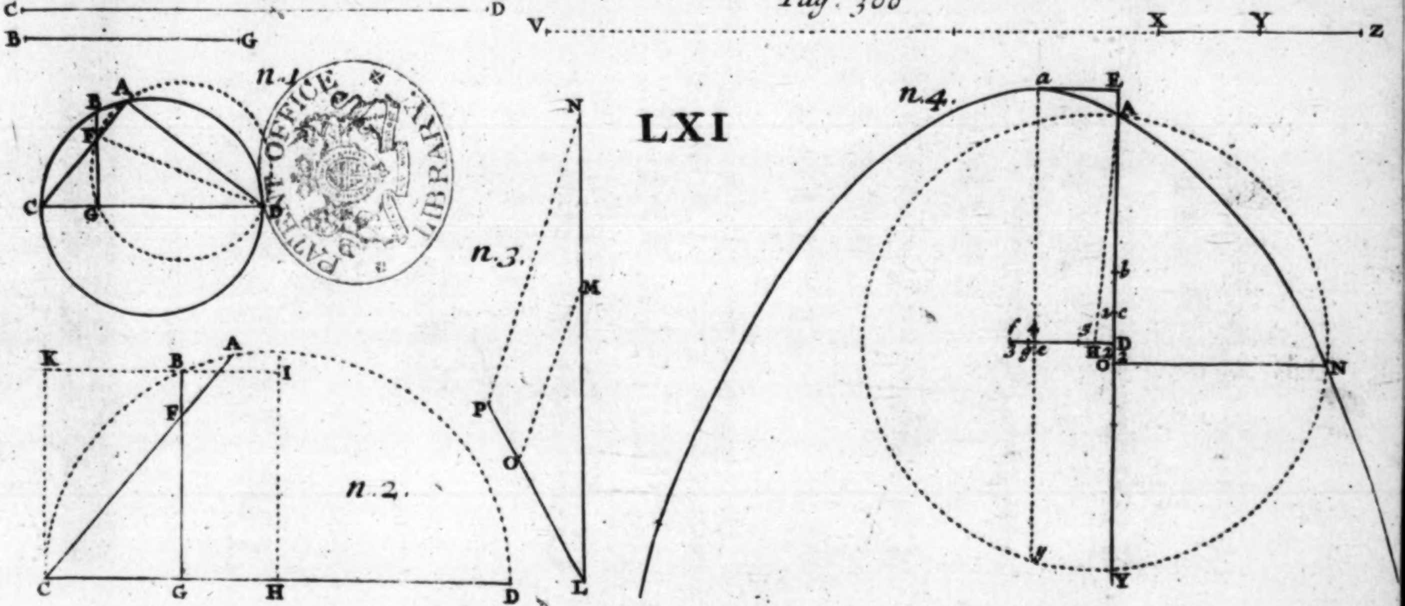

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